EXTENSION PROBLEM AND HARNACK'S INEQUALITY FOR SOME FRACTIONAL OPERATORS

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ABSTRACT. The fractional Laplacian can be obtained as a Dirichlet-to-Neumann map via an extension problem to the upper half space. In this paper we prove the same type of characterization for the fractional powers of second order partial differential operators in some class. We also get a Poisson formula and a system of Cauchy-Riemann equations for the extension. The method is applied to the fractional harmonic oscillator $H^\sigma = (-\Delta + |x|^2)^\sigma$ to deduce a Harnack's inequality. A pointwise formula for $H^\sigma f(x)$ and some maximum and comparison principles are derived.

1. INTRODUCTION

In the last years there has been a growing interest in the study of nonlinear problems involving fractional powers of the Laplace operator $(-\Delta)^\sigma$, $0 < \sigma < 1$. The fractional Laplacian of a function $f : \mathbb{R}^n \to \mathbb{R}$ is defined via Fourier transform as

$$\widehat{(-\Delta)^\sigma f}(\xi) = |\xi|^{2\sigma}\, \widehat{f}(\xi), \tag{1.1}$$

and it can be expressed by the pointwise formula

$$(-\Delta)^\sigma f(x) = c_{n,\sigma}\,\mathrm{P.V.}\int_{\mathbb{R}^n} \frac{f(x)-f(z)}{|x-z|^{n+2\sigma}}\,dz, \tag{1.2}$$

where $c_{n,\sigma}$ is a positive constant. Observe from (1.2) that the fractional Laplacian is a nonlocal operator. This fact does not allow to apply local PDE techniques to treat nonlinear problems for $(-\Delta)^\sigma$. To overcome this difficulty, L. Caffarelli and L. Silvestre showed in [2] that any fractional power of the Laplacian can be determined as an operator that maps a Dirichlet boundary condition to a Neumann-type condition via an extension problem. To be more precise, consider the function $u = u(x, y) : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ that solves the boundary value problem

$$u(x,0) = f(x), \qquad x \in \mathbb{R}^n, \tag{1.3}$$

$$\Delta_x u + \frac{1-2\sigma}{y}\,u_y + u_{yy} = 0, \qquad x \in \mathbb{R}^n,\ y > 0. \tag{1.4}$$

Then, up to a multiplicative constant depending only on σ,

$$-\lim_{y\to 0^+} y^{1-2\sigma} u_y(x,y) = (-\Delta)^\sigma f(x).$$

This characterization of $(-\Delta)^\sigma f$ via the local (degenerate) PDE (1.4) was used for the first time in [1] to get regularity estimates for the obstacle problem for the fractional Laplacian.

To solve (1.3)-(1.4), Caffarelli and Silvestre noted that (1.4) can be though as the harmonic extension of f in $2 - 2\sigma$ dimensions more (see [2]). From there, they established the fundamental solution and, using a conjugate equation, a Poisson formula for u. Furthermore, taking advantage of the general theory of degenerate elliptic equations developed by Fabes, Jerison, Kenig and Serapioni in 1982-83, they proved Harnack's estimates for u (and thus for f).

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Let Ω be an open subset of $\mathbb{R}^n$, $n \geq 1$, and let $d\eta$ be a positive measure defined on Ω. Consider a linear second order partial differential operator L, that we assume to be nonnegative, densely defined, and self-adjoint in $L^2(\Omega, d\eta)$. The fractional powers L^σ, $0 < \sigma < 1$, can be defined in a spectral way, see Section 2.

The aim of this paper is to describe any fractional power L^σ as an operator that maps a Dirichlet condition to a Neumann-type condition via an extension problem as in [2], developing also the corresponding properties (Poisson formula, fundamental solution, conjugate equation, Cauchy-Riemann equations). With this characterization, the interior Harnack's inequality for any fractional power of one of the most basic Schrödinger operators, the harmonic oscillator $H = -\Delta + |x|^2$, is consequently deduced. Besides, we find an explicit pointwise expression for the nonlocal operator H^σ that will allow us to get some maximum and comparison principles.

Fractional operators appear in physics, when considering fractional kinetics and anomalous transport [14].

The heat-diffusion semigroup $\left\{e^{-tL}\right\}_{t\geq 0}$ generated by L will play a crucial role in our work.

Our first main result is the following.

Theorem 1.1. *Let $f \in \operatorname{Dom}(L^\sigma)$. A solution of the extension problem*

$$u(x,0) = f(x), \qquad \text{on } \Omega; \tag{1.5}$$

$$-L_x u + \frac{1-2\sigma}{y}\, u_y + u_{yy} = 0, \qquad \text{in } \Omega \times (0,\infty); \tag{1.6}$$

is given by

$$u(x,y) = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-tL}(L^\sigma f)(x) e^{-\frac{y^2}{4t}}\, \frac{dt}{t^{1-\sigma}}, \tag{1.7}$$

and

$$\lim_{y\to 0^+} \frac{u(x,y)-u(x,0)}{y^{2\sigma}} = \frac{\Gamma(-\sigma)}{4^\sigma \Gamma(\sigma)} L^\sigma f(x) = \frac{1}{2\sigma} \lim_{y\to 0^+} y^{1-2\sigma} u_y(x,y). \tag{1.8}$$

Moreover, the following Poisson formula for u holds:

$$u(x,y) = \frac{y^{2\sigma}}{4^\sigma \Gamma(\sigma)} \int_0^\infty e^{-tL} f(x) e^{-\frac{y^2}{4t}}\, \frac{dt}{t^{1+\sigma}} = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-\frac{y^2}{4r}L} f(x) e^{-r}\, \frac{dr}{r^{1-\sigma}}. \tag{1.9}$$

All identities in Theorem 1.1 are understood in $L^2(\Omega, d\eta)$. Note that a solution u to the degenerate boundary value problem (1.5)-(1.6) is written explicitly in terms of the heat semigroup e^{-tL} acting on $L^\sigma f$. From here, the Poisson formula (1.9) can be immediately obtained (see the proof in Section 2), where no fractional power of L is involved. When $L = -\Delta$, the extension result of [2] is recovered (see Examples 2.14). More properties concerning the Poisson formula are contained in Theorem 2.1. Moreover, (1.9) can be derived as in [2] (Remark 2.6): use the fundamental solution (that involves the kernel of the heat semigroup generated by L) and an appropriate conjugate equation (2.10) to infer the Poisson kernel (see (2.11)). The conjugate equation will be studied in detail by defining Cauchy-Riemann equations (2.12) adapted to equation (1.6). See Section 2.

If L has discrete spectrum, i.e. $L\phi_k = \lambda_k \phi_k$, $\lambda_k \geq 0$, and $\{\phi_k\}_{k\in\mathbb{N}_0}$ is an orthonormal basis of $L^2(\Omega, d\eta)$, the definition of the fractional power L^σ is given in the natural way: if $f \in L^2(\Omega, d\eta)$ has the property that $\sum_k \lambda_k^{2\sigma} |\langle f, \phi_k\rangle|^2 = \sum_k \lambda_k^{2\sigma} \left|\int_\Omega f\phi_k\, d\eta\right|^2 < \infty$, then

$$L^\sigma f = \sum_k \lambda_k^\sigma \langle f, \phi_k\rangle \phi_k, \qquad \text{sum in } L^2(\Omega). \tag{1.10}$$

In Section 3 it is shown that, under this assumption, (1.5)-(1.6) has a unique solution u (vanishing as $y \to \infty$) such that (1.8) holds in the $L^2(\Omega)$-sense. The proof is elementary using orthogonal expansions: just write $u(x,y) = \sum_k c_k(y)\phi_k(x)$ and observe that the coefficients c_k satisfy a Bessel equation. Hence, for the existence and uniqueness in this case, the general theory of degenerate PDE's mentioned above is not needed. This method also gives us local Neumann solutions (see Subsection 3.2).

Let us now turn to the case of the fractional harmonic oscillator. We will be able to define $H^\sigma f$ for all tempered distributions f. If f is a function that has also some local regularity then the extension result is true in the classical sense (Theorem 4.2 and Remark 4.3). This last fact is an essential ingredient for the second main result of this article: the interior Harnack's inequality for H^σ.

Theorem 1.2. *Let $x_0 \in \mathbb{R}^n$ and $R > 0$. Then there exists a positive constant C depending only on n, σ, x_0 and R such that*

$$\sup_{B_{R/2}(x_0)} f \leq C \inf_{B_{R/2}(x_0)} f,$$

for all nonnegative functions $f : \mathbb{R}^n \to \mathbb{R}$ that are C^2 in $B_R(x_0)$ and such that $H^\sigma f(x) = 0$ for all $x \in B_R(x_0)$.

The Harnack's inequality is valid for $0 < \sigma < 1$ and the proof given in Section 4 is based (as we already remarked) on Theorem 4.2 and the Harnack's inequality for degenerate Schrödinger operators proved by C. E. Gutiérrez in [4] (this idea is contained in [2] for the case of the fractional Laplacian). The Harnack's inequality for H ($\sigma = 1$) follows from general results (see [13]).

The final part of the paper is devoted to the study of the pointwise expression of the fractional harmonic oscillator and some of its consequences. To that end we collect some previous facts about the fractional Laplacian $(-\Delta)^\sigma$. The natural way to arrive to (1.2) starting from (1.1) would be by taking the inverse Fourier transform. However, this path can be avoided if we consider the classical formula for L^σ that involves the heat-diffusion semigroup generated by L:

$$L^\sigma f(x) = \frac{1}{\Gamma(-\sigma)} \int_0^\infty \left(e^{-tL} f(x) - f(x) \right) \frac{dt}{t^{1+\sigma}}. \tag{1.11}$$

Note that (1.11) is motivated by the identity $\lambda^\sigma = \frac{1}{\Gamma(-\sigma)} \int_0^\infty (e^{-t\lambda} - 1) \frac{dt}{t^{1+\sigma}}$, $\lambda > 0$. When $L = -\Delta$ and $f \in \mathcal{S}$ in (1.11), the Fourier transform recovers (1.1). Furthermore, the formula allows us to obtain (in a very simple way) expression (1.2) with the constant $c_{n,\sigma}$ computed explicitly and in particular to see (Proposition 5.3) that if a function f is C^2 around some $x \in \mathbb{R}^n$ then

$$\lim_{\sigma \to 1^-} (-\Delta)^\sigma f(x) = -\Delta f(x).$$

In Section 5 we put $L = H$ in (1.11) to get a pointwise formula for $H^\sigma f(x)$ (see Theorem 5.7) and, from there, some maximum and comparison principles for H^σ.

Throughout this paper $\mathcal{S}$ is the Schwartz class of rapidly decreasing $C^\infty(\mathbb{R}^n)$ functions, the letter C denotes a constant that may change in each occurrence and it will depend on the parameters involved (whenever it is necessary we point out this dependence with subscripts) and Γ stands for the Gamma function. We restrict our attention to $0 < \sigma < 1$ and, in this range, $\Gamma(-\sigma) := \frac{\Gamma(1-\sigma)}{-\sigma} < 0$.

2. The extension problem

We begin with the basics of the spectral analysis that will be used throughout this Section. The complete details can be found in [8, Ch. 12 and 13]. Since L is a nonnegative, densely defined and self-adjoint operator on $L^2(\Omega, d\eta) = L^2(\Omega)$, there is a unique resolution E of the identity, supported on the spectrum of L (which is a subset of $[0, \infty)$), such that

$$L = \int_0^\infty \lambda \, dE(\lambda).$$

The identity above is a shorthand notation that means

$$\langle Lf, g \rangle_{L^2(\Omega)} = \int_0^\infty \lambda \, dE_{f,g}(\lambda), \qquad f \in \mathrm{Dom}(L), \; g \in L^2(\Omega),$$

where $dE_{f,g}(\lambda)$ is a regular Borel complex measure of bounded variation concentrated on the spectrum of L, with $d\left|E_{f,g}\right|(0, \infty) \leq \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}$. If $\phi(\lambda)$ is a real measurable function defined on $[0, \infty)$, then the operator $\phi(L)$ is given formally by

$$\phi(L) = \int_0^\infty \phi(\lambda) \, dE(\lambda). \tag{2.1}$$

That is, $\phi(L)$ is the operator with domain

$$\mathrm{Dom}(\phi(L))=\left\{f\in L^2(\Omega):\int_0^\infty|\phi(\lambda)|^2\,dE_{f,f}(\lambda)<\infty\right\},$$

defined by

$$\langle\phi(L)f,g\rangle_{L^2(\Omega)}=\left\langle\int_0^\infty\phi(\lambda)\,dE(\lambda)f,g\right\rangle_{L^2(\Omega)}=\int_0^\infty\phi(\lambda)\,dE_{f,g}(\lambda). \tag{2.2}$$

These considerations allow us to define the following operators:

The heat-diffusion semigroup generated by L: with domain $L^2(\Omega)$,

$$e^{-tL}=\int_0^\infty e^{-t\lambda}\,dE(\lambda),\qquad t\geq 0.$$

We have the contraction property in $L^2(\Omega)$: $\left\|e^{-tL}f\right\|_{L^2(\Omega)}\leq\|f\|_{L^2(\Omega)}$.

The fractional operators L^σ, for $0<\sigma<1$: with domain $\mathrm{Dom}(L^\sigma)\subset\mathrm{Dom}(L)$,

$$L^\sigma=\int_0^\infty\lambda^\sigma\,dE(\lambda)=\frac{1}{\Gamma(-\sigma)}\int_0^\infty\left(e^{-tL}-\mathrm{Id}\right)\,\frac{dt}{t^{1+\sigma}}.$$

The negative powers $L^{-\sigma}$, for $\sigma>0$:

$$L^{-\sigma}=\int_0^\infty\lambda^{-\sigma}\,dE(\lambda)=\frac{1}{\Gamma(\sigma)}\int_0^\infty e^{-tL}\,\frac{dt}{t^{1-\sigma}}. \tag{2.3}$$

Proof of Theorem 1.1.

1. First we prove that $u(\cdot,y)\in L^2(\Omega)$ and, for all $g\in L^2(\Omega)$,

$$\langle u(\cdot,y),g(\cdot)\rangle_{L^2}=\frac{1}{\Gamma(\sigma)}\int_0^\infty\left\langle e^{-tL}(L^\sigma f),g\right\rangle_{L^2(\Omega)}e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}}. \tag{2.4}$$

For each $R>0$ we let

$$u_R(x,y)=\frac{1}{\Gamma(\sigma)}\int_0^R e^{-tL}(L^\sigma f)(x)e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}}.$$

Since $f\in\mathrm{Dom}(L^\sigma)$, $e^{-tL}(L^\sigma f)\in L^2(\Omega)$. Moreover, $e^{-\frac{y^2}{4t}}/t^{1-\sigma}$ is integrable near 0 as a function of t. Then, using Bochner's Theorem, (2.2), the fact that $dE_{f,g}(\lambda)$ is of bounded variation, and the change of variables $t=r/\lambda$, we have

$$\begin{aligned}\langle u_R(\cdot,y),g(\cdot)\rangle_{L^2(\Omega)}&=\frac{1}{\Gamma(\sigma)}\int_0^R\left\langle e^{-tL}L^\sigma f,g\right\rangle_{L^2(\Omega)}e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}}\\&=\frac{1}{\Gamma(\sigma)}\int_0^R\int_0^\infty e^{-t\lambda}\lambda^\sigma\,dE_{f,g}(\lambda)\,e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}}\\&=\frac{1}{\Gamma(\sigma)}\int_0^\infty\int_0^R e^{-t\lambda}(t\lambda)^\sigma e^{-\frac{y^2}{4t}}\,\frac{dt}{t}\,dE_{f,g}(\lambda)\\&=\frac{1}{\Gamma(\sigma)}\int_0^\infty\int_0^{R\lambda}e^{-r}r^\sigma e^{-\frac{y^2}{4r}\lambda}\,\frac{dr}{r}\,dE_{f,g}(\lambda),\end{aligned}$$

so that

$$\left|\langle u_R(\cdot,y),g(\cdot)\rangle_{L^2(\Omega)}\right|\leq\frac{1}{\Gamma(\sigma)}\int_0^\infty\int_0^\infty e^{-r}r^\sigma\,\frac{dr}{r}\,d\left|E_{f,g}\right|(\lambda)\leq\|f\|_{L^2(\Omega)}\|g\|_{L^2(\Omega)}.$$

Therefore, for each fixed $y>0$, $u_R(\cdot,y)$ is in $L^2(\Omega)$, and $\|u_R(\cdot,y)\|_{L^2(\Omega)}\leq\|f\|_{L^2(\Omega)}$.

The last calculation shows that $\lim_{R_1,R_2\to\infty}\langle u_{R_2}(\cdot,y)-u_{R_1}(\cdot,y),g(\cdot)\rangle_{L^2(\Omega)}=0$. Then, for any sequence $\left\{R^j\right\}_{j\in\mathbb{N}}$ of positive numbers, with $R^j\nearrow\infty$, the family $\{u_{R^j}(\cdot,y)\}_{j\in\mathbb{N}}$ is a Cauchy sequence

of bounded linear operators on $L^2(\Omega)$. Thus, $u_R(\cdot,y)\to u(\cdot,y)$ weakly in $L^2(\Omega)$, as $R\to\infty$, and $u(\cdot,y)\in L^2(\Omega)$. Moreover,

$$\begin{aligned}
\langle u(\cdot,y),g(\cdot)\rangle_{L^2(\Omega)} &= \lim_{R\to\infty}\langle u_R(\cdot,y),g(\cdot)\rangle_{L^2(\Omega)} = \lim_{R\to\infty}\frac{1}{\Gamma(\sigma)}\int_0^\infty\int_0^R e^{-t\lambda}(t\lambda)^\sigma e^{-\frac{y^2}{4t}}\,\frac{dt}{t}\,dE_{f,g}(\lambda)\\
&=\frac{1}{\Gamma(\sigma)}\int_0^\infty\int_0^\infty e^{-t\lambda}(t\lambda)^\sigma e^{-\frac{y^2}{4t}}\,\frac{dt}{t}\,dE_{f,g}(\lambda)=\frac{1}{\Gamma(\sigma)}\int_0^\infty\int_0^\infty e^{-t\lambda}\lambda^\sigma\,dE_{f,g}(\lambda)\,e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}}\\
&=\frac{1}{\Gamma(\sigma)}\int_0^\infty \langle e^{-tL}(L^\sigma f),g\rangle_{L^2(\Omega)}e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}},
\end{aligned}$$

where the limit can be taken inside the integral because the double integral converges absolutely. Hence, (2.4) follows.

2. Next we show that $u(\cdot,y)\in\mathrm{Dom}(L)$, that is,

$$\lim_{s\to0^+}\left\langle\frac{e^{-sL}u(\cdot,y)-u(\cdot,y)}{s},g(\cdot)\right\rangle_{L^2(\Omega)}\quad\text{exists for all } g\in L^2(\Omega).$$

As e^{-sL} is self-adjoint, by (2.4) we have

$$\begin{aligned}
\left\langle e^{-sL}u(\cdot,y),g(\cdot)\right\rangle_{L^2(\Omega)} &= \left\langle u(\cdot,y),e^{-sL}g(\cdot)\right\rangle_{L^2(\Omega)}=\frac{1}{\Gamma(\sigma)}\int_0^\infty\left\langle e^{-tL}L^\sigma f,e^{-sL}g\right\rangle_{L^2(\Omega)}e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}}\\
&=\frac{1}{\Gamma(\sigma)}\int_0^\infty\left\langle e^{-sL}e^{-tL}L^\sigma f,g\right\rangle_{L^2(\Omega)}e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}}.
\end{aligned}$$

Hence, (2.4), (2.2), Fubini's Theorem and dominated convergence give

$$\begin{aligned}
\left\langle\frac{e^{-sL}u(\cdot,y)-u(\cdot,y)}{s},g(\cdot)\right\rangle_{L^2(\Omega)} &= \frac{1}{\Gamma(\sigma)}\int_0^\infty\left\langle\frac{e^{-sL}e^{-tL}L^\sigma f-e^{-tL}L^\sigma f}{s},g\right\rangle_{L^2(\Omega)}e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}}\\
&=\frac{1}{\Gamma(\sigma)}\int_0^\infty\int_0^\infty\frac{e^{-s\lambda}e^{-t\lambda}\lambda^\sigma-e^{-t\lambda}\lambda^\sigma}{s}\,dE_{f,g}(\lambda)\,e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}}\\
&=\frac{1}{\Gamma(\sigma)}\int_0^\infty\int_0^\infty\frac{e^{-s\lambda}e^{-t\lambda}\lambda^\sigma-e^{-t\lambda}\lambda^\sigma}{s}\,e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}}\,dE_{f,g}(\lambda)\\
&\underset{s\to0^+}{\longrightarrow}\frac{1}{\Gamma(\sigma)}\int_0^\infty\int_0^\infty\partial_t(e^{-t\lambda})\lambda^\sigma e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}}\,dE_{f,g}(\lambda)\\
&=\frac{1}{\Gamma(\sigma)}\int_0^\infty\int_0^\infty\partial_t(e^{-t\lambda})\lambda^\sigma\,dE_{f,g}(\lambda)\,e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}}\\
&=-\frac{1}{\Gamma(\sigma)}\int_0^\infty\left\langle Le^{-tL}L^\sigma f,g\right\rangle_{L^2(\Omega)}e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}}
\end{aligned}$$

3. We check the boundary condition (1.5): for $g\in L^2(\Omega)$, by (2.4),

$$\begin{aligned}
\langle u(\cdot,y),g(\cdot)\rangle &= \frac{1}{\Gamma(\sigma)}\int_0^\infty\int_0^\infty e^{-t\lambda}(t\lambda)^\sigma\,dE_{f,g}(\lambda)\,e^{-\frac{y^2}{4t}}\frac{dt}{t^{1-\sigma}}\\
&=\frac{1}{\Gamma(\sigma)}\int_0^\infty\int_0^\infty e^{-r}r^\sigma e^{-\frac{y^2\lambda}{4r}}\,\frac{dr}{r}\,dE_{f,g}(\lambda)\underset{y\to0}{\longrightarrow}\langle f,g\rangle_{L^2(\Omega)}.
\end{aligned}$$

4. The function u is differentiable with respect to y and

$$u_y(x,y)=\frac{1}{\Gamma(\sigma)}\int_0^\infty e^{-tL}(L^\sigma f)(x)\,\partial_y(e^{-\frac{y^2}{4t}})\,\frac{dt}{t^{1-\sigma}}=\frac{-1}{\Gamma(\sigma)}\int_0^\infty e^{-tL}(L^\sigma f)(x)\,\frac{ye^{-\frac{y^2}{4t}}}{2t}\,\frac{dt}{t^{1-\sigma}}. \tag{2.5}$$

Indeed applying (2.4), dominated convergence and Bochner's Theorem we get

$$\begin{aligned}\lim_{h\to 0}\left\langle \frac{u(\cdot,y+h)-u(\cdot)}{h},g(\cdot)\right\rangle_{L^2(\Omega)} &= \frac{1}{\Gamma(\sigma)}\int_0^\infty \left\langle e^{-tL}(L^\sigma f),g\right\rangle_{L^2(\Omega)}\partial_y(e^{-\frac{y^2}{4t}})\,\frac{dt}{t^{1-\sigma}}\\ &= \left\langle \frac{1}{\Gamma(\sigma)}\int_0^\infty e^{-tL}(L^\sigma f)\partial_y(e^{-\frac{y^2}{4t}})\,\frac{dt}{t^{1-\sigma}},g\right\rangle_{L^2(\Omega)}.\end{aligned}$$

5. The function u verifies the extension equation (1.6). Observe that the integral defining u_y in (2.5) is absolutely convergent as a Bochner integral, and it can be differentiated again with respect to y. Hence,

$$\begin{aligned}\left\langle \frac{1-2\sigma}{y}\,u_y(\cdot,y)+u_{yy}(\cdot,y),g(\cdot)\right\rangle_{L^2(\Omega)} &= \frac{1}{\Gamma(\sigma)}\int_0^\infty \left\langle e^{-tL}L^\sigma f,g\right\rangle_{L^2(\Omega)}\left(\frac{\sigma-1}{t}+\frac{y^2}{4t^2}\right)e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}}\\ &= -\frac{1}{\Gamma(\sigma)}\int_0^\infty \partial_t\left[\int_0^\infty e^{-t\lambda}\lambda^\sigma\,dE_{f,g}(\lambda)\right]e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}}\\ &= \frac{1}{\Gamma(\sigma)}\int_0^\infty \lambda\int_0^\infty e^{-t\lambda}\lambda^\sigma e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}}\,dE_{f,g}(\lambda)\\ &= \left\langle L\int_0^\infty e^{-tL}(L^\sigma f)e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}},g\right\rangle = \left\langle Lu(\cdot,y),g(\cdot)\right\rangle_{L^2(\Omega)}.\end{aligned}$$

6. Let us check (1.8). Note that, for all $g\in L^2(\Omega)$, by (2.4) and the change of variables $t=y^2/(4r)$,

$$\left\langle \frac{u(\cdot,y)-u(\cdot,0)}{y^{2\sigma}},g(\cdot)\right\rangle_{L^2(\Omega)} = \frac{1}{4^\sigma\Gamma(\sigma)}\int_0^\infty \left\langle e^{-\frac{y^2}{4r}L}L^\sigma f,g\right\rangle_{L^2(\Omega)}\left(\frac{e^{-r}-1}{r^\sigma}\right)\frac{dr}{r},$$

therefore, since $\lim_{t\to 0^+}\left\langle e^{-tL}L^\sigma f,g\right\rangle_{L^2(\Omega)}=\left\langle L^\sigma f,g\right\rangle_{L^2(\Omega)}$, by dominated convergence, we obtain the first identity in (1.8). Using (2.5) and the same change of variables, the second equality of (1.8) follows analogously.

7. Finally, we derive the Poisson formula (1.9). By (2.4), (2.2), Fubini's Theorem and the change of variables $t=y^2/(4r\lambda)$, we get

$$\begin{aligned}\left\langle u(\cdot,y),g(\cdot)\right\rangle_{L^2(\Omega)} &= \frac{1}{\Gamma(\sigma)}\int_0^\infty\int_0^\infty e^{-t\lambda}(t\lambda)^\sigma e^{-\frac{y^2}{4t}}\,\frac{dt}{t}\,dE_{f,g}(\lambda)\\ &= \frac{1}{\Gamma(\sigma)}\int_0^\infty\int_0^\infty e^{-\frac{y^2}{4r}}\left(\frac{y^2}{4r}\right)^\sigma e^{-r\lambda}\,\frac{dr}{r}\,dE_{f,g}(\lambda) = \frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)}\int_0^\infty \left\langle e^{-tL}f,g\right\rangle_{L^2(\Omega)}e^{-\frac{y^2}{4r}}\,\frac{dr}{r^{1+\sigma}}\\ &= \left\langle \frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)}\int_0^\infty e^{-tL}f\,e^{-\frac{y^2}{4r}}\,\frac{dr}{r^{1+\sigma}},g\right\rangle_{L^2(\Omega)}.\end{aligned}$$

The last equality is due to Bochner's Theorem.

The second identity of (1.9) follows from the first one via the change of variables $r=y^2/(4t)$. □

In what follows, we assume that the heat-diffusion semigroup is given by integration against a nonnegative *heat kernel* $K_t(x,z)$, $t>0$, $x,z\in\Omega$, that is,

$$e^{-tL}f(x)=\int_\Omega K_t(x,z)f(z)\,d\eta(z).$$

Since e^{-tL} is self-adjoint, $K_t(x,z)=K_t(z,x)$. The second assumption we make is that the heat kernel belongs to the domain of L, and $\partial_tK_t(x,z)=LK_t(x,z)$, the derivative with respect to t is understood in the classical sense. This implies that

$$\partial_t\int_\Omega K_t(x,z)f(z)\,d\eta(z)=\int_\Omega \partial_tK_t(x,z)f(z)\,d\eta(z),\qquad f\in L^2(\Omega).$$

Motivated by concrete examples, we add the hypotheses that given x, there exists a constant C_x and $\varepsilon>0$, such that $\|K_t(x,\cdot)\|_{L^2(\Omega)}+\|\partial_tK_t(x,\cdot)\|_{L^2(\Omega)}\le C_x(1+t^\varepsilon)t^{-\varepsilon}$.

Theorem 2.1 (Poisson formula)**.** *Denote by $\mathcal{P}_y^\sigma f(x)$ the function $u(x,y)$ given in* (1.9)*. Then:*

(1) *We have $\mathcal{P}_y^\sigma f(x) = \int_\Omega P_y^\sigma(x,z) f(z)\, d\eta(z)$, where the* Poisson kernel

$$P_y^\sigma(x,z) := \frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)} \int_0^\infty K_t(x,z) e^{-\frac{y^2}{4t}}\, \frac{dt}{t^{1+\sigma}}, \tag{2.6}$$

is, for each fixed $z\in\Omega$, an $L^2(\Omega)$-function that verifies (1.6)*.*

(2) $\sup_{y\ge 0} \left|\mathcal{P}_y^\sigma f\right| \le \sup_{t\ge 0}\left|e^{-tL} f\right|$*, in Ω.*

(3) *If e^{-tL} has the contraction property in $L^p(\Omega)$, then $\left\|\mathcal{P}_y^\sigma f\right\|_{L^p(\Omega)} \le \|f\|_{L^p(\Omega)}$, for all $y\ge 0$.*

(4) *If $\lim_{t\to 0^+} e^{-tL} f = f$ in $L^p(\Omega)$, then $\lim_{y\to 0^+} \mathcal{P}_y^\sigma f = f$ in $L^p(\Omega)$.*

Proof. The integral formula in (1) can be verified by using (1.9), Bochner's and Fubini's Theorems.

In order to see that the Poisson kernel satisfies (1.6), we begin by showing that it belongs to the domain of L. By the assumptions established on the L^2-norm of the heat kernel, $P_y^\sigma(\cdot,z)\in L^2(\Omega)$, for each z, and, by Bochner's Theorem,

$$e^{-sL} P_y^\sigma(\cdot,z) = \frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)} \int_0^\infty e^{-sL} K_t(x,z) e^{-\frac{y^2}{4t}}\, \frac{dt}{t^{1+\sigma}}, \qquad s\ge 0.$$

With this, we have

$$\frac{e^{-sL}P_y^\sigma(\cdot,z) - P_y^\sigma(\cdot,z)}{s} = \frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)} \int_0^\infty \frac{e^{-sL}K_t(\cdot,z) - K_t(\cdot,z)}{s}\, e^{-\frac{y^2}{4r}} \frac{dr}{r^{1+\sigma}}. \tag{2.7}$$

We use the Mean Value Theorem, the fact that $K_t(\cdot,z)\in \mathrm{Dom}(L)$, and the contraction property of e^{-sL}, to get

$$\begin{aligned}\left\|\frac{e^{-sL}K_t(\cdot,z) - K_t(\cdot,z)}{s}\right\|_{L^2(\Omega)} &= \left\|Le^{-\theta L} K_t(\cdot,z)\right\|_{L^2(\Omega)} = \left\|e^{-\theta L} L K_t(\cdot,z)\right\|_{L^2(\Omega)} \\ &\le \|LK_t(\cdot,z)\|_{L^2(\Omega)} = \|\partial_t K_t(\cdot,z)\|_{L^2(\Omega)} \le C_z(1+t^\varepsilon)t^{-\varepsilon}.\end{aligned}$$

Hence, the Dominated Convergence Theorem (for Bochner integrals) can be applied in (2.7) to see that the limit as $s\to 0^+$ of both sides exists, and

$$-L_x P_y^\sigma(x,z) = \frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)} \int_0^\infty \partial_t\left(K_t(x,z)\right) e^{-\frac{y^2}{4t}}\, \frac{dt}{t^{1+\sigma}}.$$

Now we are in position to check that $P_y^\sigma(x,z)$ verifies (1.6). Note that, by dominated convergence, the derivatives with respect to y of $P_y^\sigma(x,z)$ exist and can be computed by differentiation inside the integral sign in (2.6). Then, using integration by parts,

$$\begin{aligned}\frac{1-2\sigma}{y}\,\partial_y P_y^\sigma(x,z) + \partial_{yy}P_y^\sigma(x,z) &= \frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)} \int_0^\infty K_t(x,z) e^{-\frac{y^2}{4t}} \left(\frac{y^2}{4t^2} - \frac{1+\sigma}{t}\right) \frac{dt}{t^{1+\sigma}} \\ &= -\frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)} \int_0^\infty \partial_t(K_t(x,z)) e^{-\frac{y^2}{4t}}\, \frac{dt}{t^{1+\sigma}} = L_x P_y^\sigma(x,z),\end{aligned}$$

thus (1) is proved. (2) follows from the second identity of (1.9). The contraction property of the heat semigroup gives (3):

$$\left\|\mathcal{P}_y^\sigma f\right\|_{L^p(\Omega)} \le \frac{1}{\Gamma(\sigma)} \int_0^\infty \left\|e^{-\frac{y^2}{4r}L} f\right\|_{L^p(\Omega)} e^{-r}\, \frac{dr}{r^{1-\sigma}} \le \|f\|_{L^p(\Omega)}\,.$$

Finally, observe that

$$\left\|\mathcal{P}_y^\sigma f - f\right\|_{L^p(\Omega)} \le \frac{1}{\Gamma(\sigma)} \int_0^\infty \left\|e^{-\frac{y^2}{4t}L} f - f\right\|_{L^p(\Omega)} e^{-r}\, \frac{dr}{r^{1-\sigma}},$$

so (4) follows. □

Remark 2.2. Note in (1.9) that, when $\sigma = 1/2$, $\mathcal{P}_y^{1/2} f = e^{-t\sqrt{L}} f$ is the classical subordinated Poisson semigroup of L acting on f (see [11, p. 47 and 49]).

Proposition 2.3 (Fundamental solution of (1.6))**.** *The function*

$$\Psi_x^\sigma(z,y)=\frac{1}{\Gamma(\sigma)}\int_0^\infty K_t(x,z)e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}}, \tag{2.8}$$

satisfies equation (1.6)*,* $\Psi_x^\sigma(z,y)=\Psi_z^\sigma(x,y)$*, and*

$$\lim_{y\to0^+}\left\langle\frac{1}{2\sigma}\,y^{1-2\sigma}\partial_y\Psi_x^\sigma(\cdot,y),f(\cdot)\right\rangle_{L^2(\Omega)}=\frac{\Gamma(-\sigma)}{4^\sigma\Gamma(\sigma)}f(x). \tag{2.9}$$

Proof. As in the proof of Theorem 1.1, it can be checked that for each x,

$$\begin{aligned}\lim_{R\to\infty}\left\langle\frac{1}{\Gamma(\sigma)}\int_0^R K_t(x,\cdot)e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}},g(\cdot)\right\rangle_{L^2(\Omega)}&=\langle\Psi_x^\sigma(\cdot,y),g(\cdot)\rangle_{L^2(\Omega)}\\&=\frac{1}{\Gamma(\sigma)}\int_0^\infty e^{-tL}g(x)e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}},\end{aligned}$$

and that (2.8) satisfies (1.6). Differentiation with respect to y inside the integral in (2.8) can be performed to get

$$\frac{y^{1-2\sigma}}{2\sigma}\,\partial_y\Psi_x^\sigma(z,y)=\frac{-1}{4^\sigma\sigma\Gamma(\sigma)}\int_0^\infty K_t(x,z)e^{-\frac{y^2}{4t}}\left(\frac{y^2}{4t}\right)^{1-\sigma}\frac{dt}{t}=\frac{-1}{4^\sigma\sigma\Gamma(\sigma)}\int_0^\infty K_{\frac{y^2}{4r}}(x,z)e^{-r}\,\frac{dr}{r^\sigma}.$$

With this we obtain (2.9):

$$\begin{aligned}\frac{y^{1-2\sigma}}{2\sigma}\int_\Omega\partial_y\Psi_x^\sigma(z,y)f(z)\,d\eta(z)&=\frac{-1}{4^\sigma\sigma\Gamma(\sigma)}\int_0^\infty e^{-\frac{y^2}{4r}}f(x)e^{-r}\,\frac{dr}{r^\sigma}\\&\to\frac{-\Gamma(1-\sigma)}{4^\sigma\sigma\Gamma(\sigma)}f(x)=\frac{\Gamma(-\sigma)}{4^\sigma\Gamma(\sigma)}f(x),\qquad y\to0^+.\end{aligned}$$

□

Remark 2.4. It can also be proved that

$$\lim_{y\to0^+}\left\langle\frac{\Psi_x^\sigma(\cdot,y)-\Psi_x^\sigma(\cdot,0)}{y^{2\sigma}},f(\cdot)\right\rangle_{L^2(\Omega)}=\frac{\Gamma(-\sigma)}{4^\sigma\Gamma(\sigma)}f(x).$$

Proposition 2.5. *Let* $v(x,y)=y^{1-2\sigma}u_y(x,y)$*, where* u *solves* (1.6)*. Then* v *is a solution of the following "conjugate equation"*

$$-Lv-\frac{1-2\sigma}{y}\,v_y+v_{yy}=0,\qquad\textit{in }\Omega\times(0,\infty). \tag{2.10}$$

Proof. The calculation is analogous to the one given in [2], with the obvious modifications, and we omit it here. □

Remark 2.6. As in [2] the fundamental solution (2.8) and the "conjugate equation" (2.10) (which coincides with the conjugate equation given in [2] when $L=-\Delta$) can help us to find the Poisson kernel (2.6). Indeed, we want to write $u(x,y)=\mathcal{P}_y^\sigma f(x)=\int_\Omega P_y^\sigma(x,z)f(z)\,d\eta(z)$ where the Poisson kernel $P_y^\sigma(x,z)$ must be a solution of (1.6) for all z and $\lim_{y\to0^+}\mathcal{P}_y^\sigma f(x)=f(x)$. The right choice would be

$$P_y^\sigma(x,z)=\frac{4^{1-\sigma}\Gamma(1-\sigma)}{\Gamma(-(1-\sigma))2(1-\sigma)}\,y^{1-2(1-\sigma)}\partial_y\Psi_x^{1-\sigma}(z,y)=C_{1-\sigma}y^{1-2(1-\sigma)}\partial_y\Psi_x^{1-\sigma}(z,y), \tag{2.11}$$

since it solves the "conjugate equation" (2.10) with $1-\sigma$ in the place of σ (thus it verifies (1.6)) and by (2.9) and the choice of $C_{1-\sigma}$,

$$\lim_{y\to0^+}C_{1-\sigma}\int_\Omega y^{1-2(1-\sigma)}\partial_y\Psi_x^{1-\sigma}(z,y)f(z)\,d\eta(z)=f(x).$$

A simple calculation shows that (2.11) coincides with (2.6).

In the following discussion we shall assume that the operator L can be factorized as $L = D_i^* D_i$, where $D_i = a_i(x_i)\partial_{x_i} + b_i(x_i)$, is a one dimensional (in the ith direction) partial differential operator, and D_i^* is the formal adjoint (with respect to $d\eta$) of D_i. See examples at the end of this Section. In this case we give a definition of n conjugate functions related to the Poisson formula for u.

Let $E_\sigma = -L + \frac{1-2\sigma}{y}\partial_y + \partial_{yy}$. Then the factorization

$$E_\sigma = -\sum_{i=1}^{n} D_i^* D_i + y^{-(1-2\sigma)}\partial_y(y^{1-2\sigma}\partial_y),$$

suggests the following definition of Cauchy-Riemann equations for a system of functions $u, v_1, \ldots, v_n : \Omega \times (0,\infty) \to \mathbb{R}$ such that $E_\sigma u = 0$:

$$\begin{cases} y^{1-2\sigma}\partial_y u = D_1^* v_1 + \cdots + D_n^* v_n, & \\ D_i u = y^{-(1-2\sigma)}\partial_y v_i, & i = 1, \ldots, n, \\ D_k v_i = D_i v_k, & i, k = 1, \ldots, n. \end{cases} \tag{2.12}$$

Proposition 2.7. *Let u be a solution of $E_\sigma u = 0$ in $\Omega \times (0,\infty)$. If $v_1, \ldots, v_n$ verify* (2.12) *then each v_i solves the ith* conjugate equation

$$E^i_{1-\sigma} v_i = -L v_i + [D_i^*, D_i] v_i - \frac{1-2\sigma}{y}\,\partial_y v_i + \partial_{yy} v_i = 0, \qquad i = 1, \ldots, n, \tag{2.13}$$

where $[D_i^*, D_i] = D_i^* D_i - D_i D_i^*$.

Proof.

$$\begin{aligned} -Lv_i + [D_i^*, D_i]v_i &= -\sum_{k\neq i} D_k^* D_k v_i - D_i D_i^* v_i = -\sum_{k\neq i} D_k^* D_i v_k - D_i D_i^* v_i = -D_i\left(\sum_{k=1}^{n} D_k^* v_k\right) \\ &= -D_i\left(y^{1-2\sigma}\partial_y u\right) = -y^{1-2\sigma}\partial_y(D_i u) = -y^{1-2\sigma}\partial_y(y^{-(1-2\sigma)}\partial_y v_i) \\ &= \frac{1-2\sigma}{y}\,\partial_y v_i - \partial_{yy} v_i. \end{aligned}$$

□

Remark 2.8. The ith conjugate equation (2.13) is not the same as the "conjugate equation" (2.10). They will coincide only when $[D_i^*, D_i] = 0$. This is the case if $L = -\Delta$: the conjugate equation established in [2] is equal to each ith conjugate equation (2.13).

Proposition 2.9. *Fix $z \in \Omega$ and choose $u(x,y) = P_y^\sigma(x,z)$. Then a solution to* (2.12) *is given by the n* conjugate Poisson kernels *defined by*

$$v_i(x,y) := Q_y^{\sigma,i}(x,z) = \frac{-2}{4^\sigma\Gamma(\sigma)} D_i \int_0^\infty K_t(x,z) e^{-\frac{y^2}{4t}}\,\frac{dt}{t^\sigma}, \qquad i = 1, \ldots, n. \tag{2.14}$$

Proof. From (2.11) and the second equation of (2.12) we have $C_{1-\sigma} D_i \partial_y \Psi_x^{1-\sigma}(z,y) = \partial_y Q_y^{\sigma,i}(x,z)$, so, in view of (2.8), $Q_y^{\sigma,i}(x,z)$ can be chosen as in (2.14). Clearly $D_k Q_y^{\sigma,i}(x,z) = D_i Q_y^{\sigma,k}(x,z)$. Moreover, the first equation of (2.12) also holds:

$$\begin{aligned} D_1^* Q_y^{\sigma,1}(x,z) + \cdots + D_n^* Q_y^{\sigma,n}(x,z) &= \frac{-2}{4^\sigma\Gamma(\sigma)} \sum_{i=1}^{n} D_i^* D_i \int_0^\infty K_t(x,z) e^{-\frac{y^2}{4t}}\,\frac{dt}{t^\sigma} \\ &= \frac{-2}{4^\sigma\Gamma(\sigma)} \int_0^\infty L K_t(x,z) e^{-\frac{y^2}{4t}}\,\frac{dt}{t^\sigma} \\ &= \frac{2}{4^\sigma\Gamma(\sigma)} \int_0^\infty \partial_t K_t(x,z) e^{-\frac{y^2}{4t}}\,\frac{dt}{t^\sigma} \\ &= \frac{-2}{4^\sigma\Gamma(\sigma)} \int_0^\infty K_t(x,z) e^{-\frac{y^2}{4t}} \left(\frac{y^2}{4t} - \sigma\right) \frac{dt}{t^{1+\sigma}} \\ &= y^{1-2\sigma}\partial_y P_y^\sigma(x,z). \end{aligned}$$

□

Corollary 2.10. *The* Poisson integral of f, $u(x,y)=\mathcal{P}_y^\sigma f(x)$, *and the* n conjugate Poisson integrals *of* f *defined by*

$$v_i(x,y)\equiv \mathcal{Q}_y^{\sigma,i}f(x):=\int_\Omega Q_y^{\sigma,i}(x,z)f(z)\,d\eta(z)=\frac{-2}{4^\sigma\Gamma(\sigma)}D_i\int_0^\infty e^{-tL}f(x)e^{-\frac{y^2}{4t}}\,\frac{dt}{t^\sigma}, \tag{2.15}$$

for $i=1,\dots,n$, *solve* (2.12).

Remark 2.11. When $\sigma=1/2$, $\mathcal{Q}_y^{1/2,i}f(x)$ is the ith conjugate function of f associated to L, see [11] and [12]. A natural question arises: what is the limit of $\mathcal{Q}_y^{\sigma,i}f(x)$ as $y\to 0^+$? The answer is contained in the next result.

Theorem 2.12. *For each* $x\in\Omega$,

$$\lim_{y\to0^+}\mathcal{Q}_y^{\sigma,i}f(x)=\frac{-2\Gamma(1-\sigma)}{4^\sigma\Gamma(\sigma)}D_iL^{-(1-\sigma)}f(x).$$

Proof. From the expression of $\mathcal{Q}_y^{\sigma,i}f(x)$ in (2.15) and (2.3),

$$\lim_{y\to0^+}\mathcal{Q}_y^{\sigma,i}f(x)=\frac{-2\Gamma(1-\sigma)}{4^\sigma\Gamma(\sigma)}D_i\frac{1}{\Gamma(1-\sigma)}\int_0^\infty e^{-tL}f(x)\frac{dt}{t^{1-(1-\sigma)}}=\frac{-2\Gamma(1-\sigma)}{4^\sigma\Gamma(\sigma)}D_iL^{-(1-\sigma)}f(x).$$

□

Remark 2.13. The conclusion of Theorem 2.12 can also be obtained from the following observation: except for a multiplicative constant, the last formula of (2.15) is just the D_i-derivative of the solution of the extension problem (1.6) for $L^{1-\sigma}$ with boundary value $L^{-(1-\sigma)}f(x)$ (see (1.7)). For $\sigma=1/2$, Theorem 2.12 establishes the boundary convergence to the Riesz transforms $D_iL^{-1/2}$ (which in case $L=-\Delta$ are the classical Riesz transforms $\partial_{x_i}(-\Delta)^{-1/2}$). See [11] and [12].

Examples 2.14. *We present now some examples of operators L for which our results apply.*

The Laplacian in $\mathbb{R}^n$: *Observe that, when* $f\in\mathcal{S}$,

$$e^{t\Delta}f(x)=\int_{\mathbb{R}^n}W_t(x-z)f(z)\,dz,\quad W_t(x)=\frac{1}{(4\pi t)^{n/2}}\,e^{-\frac{|x|^2}{4t}}. \tag{2.16}$$

The Poisson formula given in [2] *is recovered: use the change of variables* $\frac{|x-z|^2+y^2}{4t}=r$, *in* (2.6), *to see that the Poisson kernel in this case is*

$$P_y^{\sigma,-\Delta}(x,z)=\frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)}\int_0^\infty\frac{e^{-\frac{|x-z|^2+y^2}{4t}}}{(4\pi t)^{n/2}}\,\frac{dt}{t^{1+\sigma}}=\frac{\Gamma(n/2+\sigma)}{\pi^{n/2}\Gamma(\sigma)}\cdot\frac{y^{2\sigma}}{\left(|x-z|^2+y^2\right)^{\frac{n+2\sigma}{2}}}.$$

The function $P_y^{1/2,-\Delta}(x,z)$ *is the classical Poisson kernel for the harmonic extension of a function to the upper half space. The Cauchy-Riemann equations read*

$$\begin{cases} y^{1-2\sigma}\partial_yu=-\left(\partial_{x_1}v_1+\cdots+\partial_{x_n}v_n\right), & \\ \qquad\partial_{x_i}u=y^{-(1-2\sigma)}\partial_yv_i, & i=1,\dots,n,\\ \qquad\partial_{x_k}v_i=\partial_{x_i}v_k, & i,k=1,\dots,n.\end{cases} \tag{2.17}$$

The case $\sigma=1/2$ *is the classical Cauchy-Riemann system for the* n *conjugate harmonic functions to* u. *In dimension one* (2.17) *reduces to*

$$\begin{cases} y^{1-2\sigma}\partial_yu=-\partial_xv,\\ \qquad\partial_xu=y^{-(1-2\sigma)}\partial_yv,\end{cases}$$

which already appeared in [7].

Classical expansions: *L can be each one of the operators arising in orthogonal expansions, like the Ornstein-Uhlenbeck operator (Hermite polynomials and Gaussian measure $d\eta(x) = e^{-|x|^2}\,dx$),*

$$-\Delta + 2x\cdot\nabla = \sum_i \left(-\partial_{x_i} + 2x_i\right)\left(\partial_{x_i}\right);$$

the harmonic oscillator (Hermite functions and Lebesgue measure $d\eta(x) = dx$),

$$-\Delta + |x|^2 = \frac{1}{2}\sum_i \left[\left(-\partial_{x_i} + x_i\right)\left(\partial_{x_i} + x_i\right) + \left(\partial_{x_i} + x_i\right)\left(-\partial_{x_i} + x_i\right)\right];$$

the Laguerre operator (Laguerre polynomials and measure $d\eta(x) = \prod_i x_i^{\alpha_i} e^{-x_i}$)

$$\sum_i x_i\partial^2_{x_i,x_i} + (\alpha_i + 1 - x_i)\partial_{x_i} = \sum_i \sqrt{x_i}\left(\partial_{x_i} + \left(\frac{\alpha_i + 1/2}{x_i} - 1\right)\right)\sqrt{x_i}\ \partial_{x_i};$$

Jacobi and ultraspherical on $(-1,1)$; etc.

We would like to point out that in these cases, due to the existence of smooth eigenfunctions, the proof of Theorem 1.1 can be performed as an exercise of convergence of orthogonal systems, and it makes it technically simpler.

Elliptic operators: *Let L be a positive self-adjoint linear elliptic partial differential operator on $L^2(\Omega)$, with Dirichlet boundary conditions, and bounded measurable coefficients. Then the heat kernel exists, and it verifies our assumptions stated before Theorem 2.1. Even more, its heat kernel has Gaussian bounds* [3, p. 89]. *We can also consider Schrödinger operators with nonnegative potentials in a large class* [3, Section 4.5].

3. Existence and uniqueness results for the extension problem

In this section we derive the concrete solution of the extension problem in the case of discrete spectrum. We also find solutions with null Neumann condition. This is done by using classical Fourier's method.

Let $\{\phi_k\}_{k\in\mathbb{N}_0}$ be an orthonormal basis of $L^2(\Omega)$ such that $L\phi_k = \lambda_k\phi_k$, $\lambda_k \geq 0$. Recall the definition of L^σ given in (1.10).

3.1. **L^2 theory.** Let $f \in L^2(\Omega)$ and look for solutions u to (1.5)-(1.6) of the form

$$u(x,y) = \sum_k c_k(y)\phi_k(x). \tag{3.1}$$

Then for each $k \geq 0$ we have to solve the following ordinary differential equation:

$$-\lambda_k c_k + \frac{1-2\sigma}{y}\,c_k' + c_k'' = 0, \qquad y > 0,$$

with initial condition $c_k(0) = \langle f, \phi_k\rangle$. According to [6, p. 106], this last equation has a general solution of the form

$$c_k(y) = y^\sigma Z_\sigma(\pm i\lambda_k^{1/2}y), \tag{3.2}$$

where Z_σ is a linear combination of Bessel functions of order σ. To have uniqueness of the solution include the boundary condition $\lim_{y\to\infty} u(x,y) = 0$, weakly in $L^2(\Omega)$, which translates to the coefficients as

$$\lim_{y\to\infty} c_k(y) = 0. \tag{3.3}$$

From [6, p. 104], Z_σ can be written as

$$Z_\sigma(z) = A_1J_\sigma(z) + A_2H_\sigma^{(1)}(z) = B_1J_\sigma(z) + B_2H_\sigma^{(2)}(z) \tag{3.4}$$

$$= C_1J_\sigma(z) + C_2J_{-\sigma}(z) = D_1H_\sigma^{(1)}(z) + D_2H_\sigma^{(2)}(z), \tag{3.5}$$

where J_σ denotes the Bessel function of the first kind and $H_\sigma^{(1)}$ and $H_\sigma^{(2)}$ are the Hankel functions. To fulfill condition (3.3) we need to review the asymptotic behavior of the Bessel functions. When $|\arg z| \le \pi - \delta$,

$$
\begin{aligned}
J_\sigma(z) &= \left(\frac{2}{\pi z}\right)^{1/2}\left[\cos\left(z-\frac{2\sigma\pi+\pi}{4}\right)\left(1+O(|z|^{-2})\right)\right. \\
&\qquad\left. -\sin\left(z-\frac{2\sigma\pi+\pi}{4}\right)\left(\frac{4\sigma^2-1}{8z}+O(|z|^{-3})\right)\right], \\
H_\sigma^{(1)}(z) &= \left(\frac{2}{\pi z}\right)^{1/2} e^{i\left(z-\frac{2\sigma\pi+\pi}{4}\right)}\left(1+O(|z|^{-1})\right), \\
H_\sigma^{(2)}(z) &= \left(\frac{2}{\pi z}\right)^{1/2} e^{-i\left(z-\frac{2\sigma\pi+\pi}{4}\right)}\left(1+O(|z|^{-1})\right).
\end{aligned}
\tag{3.6}
$$

Note that for purely imaginary z, $J_\sigma(z)\to\infty$ exponentially and $H_\sigma^{(1)}(z)\to 0$ or ∞ depending on the sign of the imaginary part of z. Putting $z = i\lambda_k^{1/2}y$ in (3.2) we see that the only possible choice as solution is the first linear combination of (3.4) as soon as $A_1 = 0$: $c_k(y) = A_{2,k}y^\sigma H_\sigma^{(1)}(i\lambda_k^{1/2}y)$. If K_σ denotes the modified Bessel function of the third kind then $H_\sigma^{(1)}(iz) = 2\pi^{-1}i^{-\sigma-1}K_\sigma(z)$ and

$$c_k(y) = A_{2,k}y^\sigma\frac{2i^{-\sigma-1}}{\pi}\,K_\sigma(\lambda_k^{1/2}y).$$

To determine $A_{2,k}$ use the initial condition. The asymptotic behavior of $K_\sigma(z)$ as $z\to 0$ reads

$$K_\sigma(z) \approx \Gamma(\sigma)2^{\sigma-1}\frac{1}{z^\sigma}. \tag{3.7}$$

So that, when $y\to 0$, $c_k(y)\approx A_{2,k}2^\sigma\pi^{-1}i^{-\sigma-1}\Gamma(\sigma)\lambda_k^{-\sigma/2}$. Therefore

$$A_{2,k} = \frac{\pi i^{1+\sigma}}{2^\sigma\Gamma(\sigma)}\lambda_k^{\sigma/2}\langle f,\phi_k\rangle.$$

Thus

$$c_k(y) = y^\sigma\frac{2^{1-\sigma}}{\Gamma(\sigma)}\lambda_k^{\sigma/2}\langle f,\phi_k\rangle K_\sigma(\lambda_k^{1/2}y). \tag{3.8}$$

Since as $|z|\to\infty$,

$$K_\sigma(z)\approx\left(\frac{\pi}{2z}\right)^{1/2}e^{-z}\left(1+O(|z|^{-1})\right),$$

the series in (3.1), with c_k as in (3.8), converges in $L^2(\Omega)$ for each $y\in(0,\infty)$. Finally, (3.7) implies that (1.5) is fulfilled in the $L^2(\Omega)$ sense.

On the other hand, by using the properties of the derivatives of K_σ (see [6, p. 110]) and (3.7), as $y\to 0$ we have

$$
\begin{aligned}
\frac{1}{2\sigma}y^{1-2\sigma}c_k'(y) &= \frac{1}{2\sigma}y^{1-2\sigma}\frac{2^{1-\sigma}}{\Gamma(\sigma)}\langle f,\phi_k\rangle\frac{d}{d(\lambda_k^{1/2}y)}\left[(\lambda_k^{1/2}y)^\sigma K_\sigma(\lambda_k^{1/2}y)\right]\frac{d(\lambda_k^{1/2}y)}{y} \\
&= \frac{1}{2\sigma}y^{1-2\sigma}\frac{2^{1-\sigma}}{\Gamma(\sigma)}\langle f,\phi_k\rangle(-1)(\lambda_k^{1/2}y)^\sigma K_{\sigma-1}(\lambda_k^{1/2}y)\lambda_k^{1/2} \\
&= \frac{2^{-\sigma}}{-\sigma\Gamma(\sigma)}\langle f,\phi_k\rangle\lambda_k^{\sigma/2}\lambda_k^{1/2}y^{1-\sigma}K_{1-\sigma}(\lambda_k^{1/2}y) \\
&\approx \frac{2^{-\sigma}}{-\sigma\Gamma(\sigma)}\langle f,\phi_k\rangle\lambda_k^{\sigma/2}\lambda_k^{1/2}y^{1-\sigma}\Gamma(1-\sigma)2^{-\sigma}\frac{1}{(\lambda_k^{1/2}y)^{1-\sigma}} \\
&= \frac{2^{-2\sigma}\Gamma(1-\sigma)}{-\sigma\Gamma(\sigma)}\lambda_k^\sigma\langle f,\phi_k\rangle = \frac{\Gamma(-\sigma)}{4^\sigma\Gamma(\sigma)}\lambda_k^\sigma\langle f,\phi_k\rangle.
\end{aligned}
$$

As a consequence,

$$\frac{1}{2\sigma}\lim_{y\to 0^+} y^{1-2\sigma}u_y(x,y) = \frac{\Gamma(-\sigma)}{4^\sigma\Gamma(\sigma)}\sum_k \lambda_k^\sigma \langle f,\phi_k\rangle\phi_k(x) = \frac{\Gamma(-\sigma)}{4^\sigma\Gamma(\sigma)}L^\sigma f(x),$$

the limit taken in the $L^2(\Omega)$-sense (see (1.10)).

3.2. **Local Neumann solutions.** Let us find a solution to (1.6) such that

$$\frac{1}{2\sigma}\lim_{y\to 0^+} y^{1-2\sigma}u_y(x,y) = 0, \qquad \text{for all } x\in\Omega. \tag{3.9}$$

Writing $u(x,y)=\sum_k d_k(y)\phi_k(x)$, condition (3.9) implies that $\lim_{y\to 0^+} y^{1-2\sigma}d_k'(y)=0$. Therefore as (see (3.2))

$$d_k'(y) = (i\lambda_k)^{1-\sigma}\frac{d}{d(i\lambda_k^{1/2}y)}\left[(i\lambda_k^{1/2}y)^\sigma Z_\sigma(i\lambda_k^{1/2}y)\right] = i\lambda_k^{1/2}y^\sigma Z_{\sigma-1}(i\lambda_k^{1/2}y),$$

we require

$$y^{1-2\sigma}d_k'(y) = i\lambda_k^{1/2}y^{1-\sigma}Z_{\sigma-1}(i\lambda_k^{1/2}y)\to 0, \qquad y\to 0.$$

When $z\to 0$ (see [6]),

$$J_\sigma(z)\approx \frac{z^\sigma}{2^\sigma\Gamma(1+\sigma)}, \quad H_\sigma^{(1)}(z)\approx \frac{2^\sigma\Gamma(\sigma)}{i\pi}\frac{1}{z^\sigma}, \quad \text{and} \quad H_\sigma^{(2)}(z)\approx -\frac{2^\sigma\Gamma(\sigma)}{i\pi}\frac{1}{z^\sigma}.$$

Then, as $y\to 0$,

$$y^{1-\sigma}J_{\sigma-1}(i\lambda_k^{1/2}y)\to \frac{(i\lambda_k^{1/2})^{\sigma-1}}{2^{\sigma-1}\Gamma(\sigma)},$$

$$y^{1-\sigma}H_{\sigma-1}^{(1)}(i\lambda_k^{1/2}y) = y^{1-\sigma}i^{2\sigma}H_{1-\sigma}^{(1)}(i\lambda_k^{1/2}y)\to \frac{2^{1-\sigma}\Gamma(1-\sigma)i^{2\sigma-1}}{\pi}(i\lambda_k^{1/2})^{1-\sigma},$$

$$y^{1-\sigma}H_{\sigma-1}^{(2)}(i\lambda_k^{1/2}y) = y^{1-\sigma}i^{-2\sigma}H_{1-\sigma}^{(2)}(i\lambda_k^{1/2}y)\to -\frac{2^{1-\sigma}\Gamma(1-\sigma)i^{-(2\sigma+1)}}{\pi}(i\lambda_k^{1/2})^{1-\sigma},$$

but

$$y^{1-\sigma}J_{1-\sigma}(i\lambda_k^{1/2}y)\approx \frac{(i\lambda_k^{1/2})^{1-\sigma}}{2^{1-\sigma}\Gamma(2-\sigma)}y^{2-2\sigma}\to 0.$$

Consequently, choose the first linear combination in (3.5) with $C_1=0$. Thus

$$d_k(y) = C_{2,k}y^\sigma J_{-\sigma}(i\lambda_k^{1/2}y),$$

verifies $\lim_{y\to 0} y^{1-2\sigma}d_k'(y)=0$. So u formally reads

$$u(x,y) = y^\sigma\sum_k C_{2,k}J_{-\sigma}(i\lambda_k^{1/2}y)\phi_k(x).$$

In order to have a convergent series (at least for small y), let us determine $C_{2,k}$. Taking into account (3.6) it is enough to fix $R>0$ and put $C_{2,k}=Ce^{-\lambda_k^{1/2}R}$.

In this way we obtained a solution u to equation (1.6) in $\Omega\times(0,R)$ that satisfies the required property (3.9).

4. The Harnack's inequality for H^σ

To prove the Harnack's inequality in Theorem 1.2 we first study the problem (1.5)-(1.6) for the harmonic oscillator $L=H=-\Delta+|x|^2$ posed in $\mathbb{R}^n$ with the Lebesgue measure $d\eta=dx$.

Fix $1\le p<\infty$ and $N>0$. Define the space

$$L^p_N = \left\{u:\mathbb{R}^n\to\mathbb{R} : \|u\|_{L^p_N} = \left(\int_{\mathbb{R}^n}\frac{|u(z)|^p}{(1+|z|^2)^{Np}}\,dz\right)^{1/p}<\infty\right\}. \tag{4.1}$$

Then $L^p_N\subset\mathcal{S}'$.

The heat semigroup generated by H (see [12]) can be given as an integral operator

$$e^{-tH}f(x)=\int_{\mathbb{R}^n}G_t(x,z)f(z)\,dz=\int_{\mathbb{R}^n}\frac{e^{-\left[\frac{1}{2}|x-z|^2\coth 2t+x\cdot z\tanh t\right]}}{(2\pi\sinh 2t)^{n/2}}\,f(z)\,dz. \tag{4.2}$$

We collect some useful facts about e^{-tH} in the next Proposition, whose proof is postponed to the end of Section 5.

Proposition 4.1. *For $f\in L^p_N$, the heat semigroup $e^{-tH}f(x)$ is well defined and*

$$\left|e^{-tH}f(x)\right|\le C\frac{(1+|x|^\rho)\,\|f\|_{L^p_N}}{t^{n/2}},\qquad x\in\mathbb{R}^n,\ t>0, \tag{4.3}$$

where $\rho>0$ depends on p and N. Moreover, $(\partial_t+H)e^{-tH}f(x)=0$ for all $x\in\mathbb{R}^n$ and $t>0$ and for $i,j=1,\dots,n$,

$$\left|\partial_{x_i}(e^{-tH}f)(x)\right|\le C\frac{(1+|x|^\rho)\,\|f\|_{L^p_N}}{t^{(n+1)/2}},\qquad \left|\partial_{x_ix_j}(e^{-tH}f)(x)\right|\le C\frac{(1+|x|^\rho)\,\|f\|_{L^p_N}}{t^{(n+2)/2}}. \tag{4.4}$$

If f is also a C^2 function in some open subset $\mathcal{O}\subset\mathbb{R}^n$ then $\lim_{t\to0}e^{-tH}f(x)=f(x)$ for all $x\in\mathcal{O}$.

In the particular case we are considering in this Section, Theorem 1.1 takes the following form, in which the relevant observation is that all identities are classical.

Theorem 4.2. *If $f\in L^p_N$ is a C^2 function in some open subset $\mathcal{O}\subset\mathbb{R}^n$ then*

$$u(x,y):=\frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)}\int_0^\infty e^{-tH}f(x)e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1+\sigma}}, \tag{4.5}$$

is well defined for all $x\in\mathbb{R}^n$, $y>0$, and

$$\begin{aligned}-H_xu+\frac{1-2\sigma}{y}\,u_y+u_{yy}&=0, &&\text{in } \mathbb{R}^n\times(0,\infty);\\ \lim_{y\to0^+}u(x,y)&=f(x), &&\text{for } x\in\mathcal{O}.\end{aligned}$$

In addition, for all $x\in\mathcal{O}$,

$$\frac{1}{2\sigma}\lim_{y\to0^+}y^{1-2\sigma}u_y(x,y)=\frac{1}{4^\sigma\Gamma(\sigma)}\int_0^\infty\left(e^{-tH}f(x)-f(x)\right)\,\frac{dt}{t^{1+\sigma}}. \tag{4.6}$$

Proof. Estimate (4.3) implies that the integral defining u is absolutely convergent and u_y and u_{yy} can be computed by taking the derivatives inside the integral sign. Moreover, by using (4.4), we have

$$H_xu(x,y)=\frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)}\int_0^\infty He^{-tH}f(x)e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1+\sigma}},$$

in the classical sense. Hence, for each $x\in\mathbb{R}^n$, u verifies the extension problem in the classical sense. To check that (4.6) is classical, we begin by recalling that

$$\int_0^\infty e^{-\frac{y^2}{4t}}\left(2\sigma-\frac{y^2}{2t}\right)\,\frac{dt}{t^{1+\sigma}}=0.$$

Thus

$$\begin{aligned}\frac{1}{2\sigma}\,y^{1-2\sigma}u_y(x,y)&=\frac{1}{2\sigma4^\sigma\Gamma(\sigma)}\int_0^\infty e^{-tH}f(x)e^{-\frac{y^2}{4t}}\left(2\sigma-\frac{y^2}{2t}\right)\,\frac{dt}{t^{1+\sigma}}\\ &=\frac{1}{2\sigma4^\sigma\Gamma(\sigma)}\int_0^\infty\left(e^{-tH}f(x)-f(x)\right)e^{-\frac{y^2}{4t}}\left(2\sigma-\frac{y^2}{2t}\right)\,\frac{dt}{t^{1+\sigma}}.\end{aligned}$$

As we shall see later (Remark 5.12) the integral in (4.6) is absolutely convergent for all $x\in\mathcal{O}$ and f as in the hypotheses. Therefore (4.6) follows. $\square$

Remark 4.3. In Section 5 we will see that for $f\in L^p_N\cap C^2(\mathcal{O})$, $H^\sigma f$ is well defined and

$$H^\sigma f(x)=\frac{1}{\Gamma(-\sigma)}\int_0^\infty\left(e^{-tH}f(x)-f(x)\right)\,\frac{dt}{t^{1+\sigma}},\qquad x\in\mathcal{O}.$$

Remark 4.4. Theorem 4.2 is valid if H is replaced by $-\Delta$ and the function f, with the same smoothness in $\mathcal{O}$, belongs to $L_\sigma := L^1_{n/2+\sigma}$. See the discussion on $(-\Delta)^\sigma$ given in Section 5.

Lemma 4.5 (Reflection extension)**.** *Fix $R > 0$ and $x_0 \in \mathbb{R}^n$. Let u be a solution of*

$$-H_x u + \frac{1-2\sigma}{y}\, u_y + u_{yy} = 0, \qquad \text{in } \mathbb{R}^n \times (0,R),$$

with

$$\lim_{y\to 0^+} y^{1-2\sigma} u_y(x,y) = 0, \qquad \text{for every } x \text{ such that } |x - x_0| < R. \tag{4.7}$$

Then the extension to $\mathbb{R}^n \times (-R,R)$ defined by

$$\tilde{u}(x,y) = \begin{cases} u(x,y), & y \geq 0; \\ u(x,-y), & y < 0; \end{cases} \tag{4.8}$$

verifies the degenerate Schrödinger equation

$$\operatorname{div}(|y|^{1-2\sigma} \nabla \tilde{u}) - |y|^{1-2\sigma} |x|^2 \tilde{u} = 0, \tag{4.9}$$

in the weak sense in $B := \left\{(x,y) \in \mathbb{R}^{n+1} : |x-x_0|^2 + y^2 < R^2\right\}$.

Proof. A nontrivial solution u can be found with the method of Subsection 3.2 since the eigenfunctions of the harmonic oscillator H are the Hermite functions h_α, $\alpha \in \mathbb{N}_0^n$, with corresponding eigenvalues $\lambda_\alpha = 2|\alpha| + n$ (see Section 5). Given $\varphi \in C_c^\infty(B)$ we want to prove that

$$I := \int_B \left(\nabla\tilde{u} \cdot \nabla\varphi + |x|^2 \tilde{u}\varphi\right) |y|^{1-2\sigma} \, dx\, dy = 0.$$

For $\delta > 0$ we have

$$\begin{aligned} I &= \int_{B\cap\{|y|\geq\delta\}} \operatorname{div}(|y|^{1-2\sigma} \varphi\nabla\tilde{u}) \, dx\, dy + \int_{B\cap\{|y|<\delta\}} \left(\nabla\tilde{u} \cdot \nabla\varphi + |x|^2 \tilde{u}\varphi\right) |y|^{1-2\sigma} \, dx\, dy \\ &= \int_{B\cap\{|y|=\delta\}} \varphi\delta^{1-2\sigma} \tilde{u}_y(x,\delta)\, dx + \int_{B\cap\{|y|<\delta\}} \left(\nabla\tilde{u} \cdot \nabla\varphi + |x|^2 \tilde{u}\varphi\right) |y|^{1-2\sigma} \, dx\, dy. \end{aligned}$$

As $\delta \to 0$, the first term above goes to zero because of (4.7) and the second term goes to zero because $\left(|\nabla\tilde{u}|^2 + |x|^2 \tilde{u}\right) |y|^{1-2\sigma}$ is a locally integrable function. □

Proof of Theorem 1.2. Let u be as in Theorem 4.2. Since f is a nonnegative function, from (4.2) and (4.5) we see that $u \geq 0$. Because of Remark 4.3, its reflection (4.8) satisfies Lemma 4.5. Note that (4.9) is a degenerate Schrödinger equation with A_2 weight $w = |y|^{1-2\sigma}$ and potential $V = |y|^{1-2\sigma} |x|^2$ such that $V/w \in L^p_w$ locally for p large enough. So we can apply the result of [4] to obtain the Harnack's inequality for $\tilde{u}$ and thus for f. □

5. Pointwise formula for H^σ and some of its consequences

The semigroup language adopted in Section 2, allows us to get the exact pointwise formula for the fractional Laplacian $(-\Delta)^\sigma$ on $\mathbb{R}^n$. The constants involved in the definition are computed exactly in an easy way.

Lemma 5.1. *For $f \in \mathcal{S}$,*

$$(-\Delta)^\sigma f(x) = \frac{1}{\Gamma(-\sigma)} \int_0^\infty \left(e^{t\Delta} f(x) - f(x)\right) \frac{dt}{t^{1+\sigma}} = \frac{4^\sigma \Gamma(n/2+\sigma)}{\pi^{n/2}\Gamma(-\sigma)} \,\mathrm{P.V.} \int_{\mathbb{R}^n} \frac{f(z) - f(x)}{|x-z|^{n+2\sigma}}\, dz. \tag{5.1}$$

Proof. The first identity follows by Fourier transform. From the fact that $e^{t\Delta}1(x) \equiv 1$ we can write

$$\int_0^\infty \left(e^{t\Delta} f(x) - f(x)\right) \frac{dt}{t^{1+\sigma}} = \int_0^\infty \int_{\mathbb{R}^n} W_t(x-z)(f(z) - f(x))\, dz\, \frac{dt}{t^{1+\sigma}} = I_1 + I_2, \tag{5.2}$$

where

$$I_1 := \int_0^\infty \int_{|x-z|>1} W_t(x-y)(f(z)-f(x))\,dz\,\frac{dt}{t^{1+\sigma}},$$

and W_t is the heat kernel for the Laplacian (2.16). Use the change of variables $s=\frac{|x-z|^2}{4t}$ to see that

$$\int_0^\infty \frac{1}{(4\pi t)^{n/2}}\,e^{-\frac{|x-z|^2}{4t}}\,\frac{dt}{t^{1+\sigma}} = \frac{4^\sigma\Gamma(n/2+\sigma)}{\pi^{n/2}}\cdot\frac{1}{|x-z|^{n+2\sigma}}. \tag{5.3}$$

So, since f is bounded, I_1 converges absolutely. Passing to polar coordinates,

$$I_2 = \int_0^\infty \frac{1}{(4\pi t)^{n/2}}\int_0^1 e^{-\frac{r^2}{4t}}r^{n-1}\int_{|z'|=1}(f(x+rz')-f(x))\,dS(z')\,dr\,\frac{dt}{t^{1+\sigma}}.$$

By Taylor's Theorem, $\int_{|z'|=1}(f(x+rz')-f(x))\,dS(z') = C_n r^2\Delta f(x)+O(r^3)$, thus

$$|I_2| \le C_{n,\Delta f(x)}\int_0^1 r^{n+1}\int_0^\infty \frac{e^{-\frac{r^2}{4t}}}{t^{n/2+\sigma}}\,\frac{dt}{t}\,dr = C_{n,\Delta f(x),\sigma}\int_0^1 r^{1-2\sigma}\,dr = C_{n,\Delta f(x),\sigma},$$

and I_2 converges. Therefore apply Fubini's Theorem in (5.2) and (5.3) to get (5.1). □

Remark 5.2. Lemma 5.1 gives the exact value of the positive constant $c_{n,\sigma}$ in (1.2). Observe that

$$c_{n,\sigma} = \frac{-4^\sigma\Gamma(n/2+\sigma)}{\pi^{n/2}\Gamma(-\sigma)} \to 0, \qquad \text{as } \sigma\to 0^+ \text{ or } \sigma\to 1^-. \tag{5.4}$$

When $f\in\mathcal{S}$ it is clear (by Fourier transform) that $\lim_{\sigma\to 1^-}(-\Delta)^\sigma f = -\Delta f$. The next Proposition shows that this is in fact valid for $f\in C^2$. Note that if $f\in\mathcal{S}$ then, from (1.1), $(-\Delta)^\sigma f\notin\mathcal{S}$, but still $(-\Delta)^\sigma f\in C^\infty$. It can be checked that for every $\beta\in\mathbb{N}_0^n$ the function $(1+|x|^{n+2\sigma})D^\beta(-\Delta)^\sigma f(x)$ is bounded. Therefore the set $L_\sigma := \left\{u:\mathbb{R}^n\to\mathbb{R} : \|u\|_{L_\sigma} = \int_{\mathbb{R}^n}\frac{|u(z)|}{1+|z|^{n+2\sigma}}\,dz<\infty\right\}$ (which is $L^1_{n/2+\sigma}$ in (4.1)), consists of all locally integrable tempered distributions u for which $(-\Delta)^\sigma u$ can be defined. If $f\in L_\sigma$ is C^2 in an open set $\mathcal{O}$ then it can be proved that $(-\Delta)^\sigma f$ is a continuous function in $\mathcal{O}$ and its values are given by the second integral in (5.1). For all the details see [9] and [10].

Proposition 5.3. *Let $f\in C^2(B_2(x))\cap L^\infty(\mathbb{R}^n)$ for some $x\in\mathbb{R}^n$. Then*

$$\lim_{\sigma\to 1^-}(-\Delta)^\sigma f(x) = -\Delta f(x).$$

Proof. Fix an arbitrary $\varepsilon>0$. Since $f\in C^2(B_2(x))$ there exists $\delta=\delta_\varepsilon>0$ such that

$$\left|D^2 f(w)-D^2 f(w')\right| < \varepsilon, \qquad \text{for all } w,w'\in\overline{B_1(x)} \text{ such that } |w-w'|<\delta. \tag{5.5}$$

Write $(-\Delta)^\sigma f(x) = c_{n,\sigma}(I+II)$ where $I=\int_{|x-z|>\delta}\frac{f(x)-f(z)}{|x-z|^{n+2\sigma}}\,dz$. We have $|I|\le C_n\sigma^{-1}\delta^{-2\sigma}\|f\|_{L^\infty}$, so that from (5.4), $c_{n,\sigma}I\to 0$ as $\sigma\to 1^-$. Using polar coordinates, Taylor's Theorem and recalling that $\int_{|z'|=1}(z_1')^2\,dS(z') = \frac{(n/2+1)\pi^{n/2}}{\Gamma(n/2+2)}$,

$$\begin{aligned}
II &= \int_0^\delta r^{-1-2\sigma}\int_{|z'|=1}(f(x)-f(x-rz'))\,dS(z')\,dr \\
&= \int_0^\delta r^{-1-2\sigma}\int_{|z'|=1} R_1 f(x,rz')\,dS(z')\,dr \\
&= \int_0^\delta r^{-1-2\sigma}\left[\frac{-\Delta f(x)(n/2+1)\pi^{n/2}r^2}{2\Gamma(n/2+2)} + \int_{|z'|=1}\left(R_1 f(x,rz') - \frac{r^2}{2}\langle D^2 f(x)z',z'\rangle\right)dS(z')\right]dr \\
&= \frac{-\Delta f(x)(n/2+1)\pi^{n/2}\delta^{2-2\sigma}}{4\Gamma(n/2+2)(1-\sigma)} + \int_0^\delta r^{-1-2\sigma}\int_{|z'|=1}\left(R_1 f(x,rz') - \frac{r^2}{2}\langle D^2 f(x)z',z'\rangle\right)dS(z')dr \\
&=: II_1 + II_2,
\end{aligned}$$

where $R_1f(x,rz')$ is the Taylor's remainder of first order. Then (5.4) entails

$$c_{n,\sigma}II_1 = \frac{-\Delta f(x)\sigma(n/2+1)\Gamma(n/2+\sigma)\delta^{2-2\sigma}}{4^{1-\sigma}\Gamma(n/2+2)\Gamma(2-\sigma)} \to -\Delta f(x)\frac{(n/2+1)\Gamma(n/2+1)}{\Gamma(n/2+2)} = -\Delta f(x),$$

as $\sigma \to 1^-$. Finally, by (5.5), $\left|R_1f(x,rz') - \frac{r^2}{2}\langle D^2f(x)z',z'\rangle\right| \le C_n r^2\varepsilon$ and $|II_2| \le C_n\delta^{2-2\sigma}(1-\sigma)^{-1}\varepsilon$. Therefore $\lim_{\sigma\to 1^{-1}} |c_{n,\sigma}II_2| \le C_n\varepsilon$. □

Remark 5.4. For $f \in C^2(B_2(x)) \cap L_\sigma$ the second identity in (5.1) is valid (the idea is to use the continuity of D^2f as in the proof of Proposition 5.3).

We shall now discuss the definition of the fractional harmonic oscillator H^σ and the pointwise formula for $H^\sigma f(x)$. The eigenfunctions of H (see [12]) are the multi-dimensional Hermite functions defined on $\mathbb{R}^n$ as $h_\alpha(x) = \Phi_\alpha(x)e^{-|x|^2/2}$, $\alpha \in \mathbb{N}_0^n$, where Φ_α are the multi-dimensional Hermite polynomials, and $Hh_\alpha = (2|\alpha|+n)h_\alpha$. Note that $h_\alpha \in \mathcal{S}$. The set of Hermite functions forms an orthonormal basis of $L^2(\mathbb{R}^n)$. Let $f \in \mathcal{S}$. The *Hermite series expansion* of f given by

$$\sum_\alpha \langle f,h_\alpha\rangle h_\alpha = \sum_{k=0}^\infty \sum_{|\alpha|=k} \langle f,h_\alpha\rangle h_\alpha, \tag{5.6}$$

with $\langle f,h_\alpha\rangle = \int_{\mathbb{R}^n} fh_\alpha\,dx$ (which converges to f in L^2), converges uniformly in $\mathbb{R}^n$ to f. This uniform convergence is a consequence of the fact that $\|h_\alpha\|_{L^\infty(\mathbb{R}^n)} \le C$ for all $\alpha \in \mathbb{N}_0^n$ and the following estimate: for every $m \in \mathbb{N}$,

$$|\langle f,h_\alpha\rangle| = \frac{|\langle H^mf,h_\alpha\rangle|}{(2|\alpha|+n)^m} \le \frac{\|H^mf\|_{L^2}}{(2|\alpha|+n)^m}, \tag{5.7}$$

since H is a symmetric operator. If $f \in \mathcal{S}$ then

$$e^{-tH}f(x) = \sum_\alpha e^{-t(2|\alpha|+n)}\langle f,h_\alpha\rangle h_\alpha(x), \qquad t \ge 0, \tag{5.8}$$

the series converging uniformly in $\mathbb{R}^n$. By the given estimates on $\|h_\alpha\|_{L^\infty}$ and $|\langle f,h_\alpha\rangle|$ the series defining the *fractional Hermite operator*

$$H^\sigma f = \sum_\alpha (2|\alpha|+n)^\sigma \langle f,h_\alpha\rangle h_\alpha \tag{5.9}$$

converges uniformly in $\mathbb{R}^n$.

Lemma 5.5. *For $f \in \mathcal{S}$,*

$$H^\sigma f(x) = \frac{1}{\Gamma(-\sigma)}\int_0^\infty \left(e^{-tH}f(x) - f(x)\right)\,\frac{dt}{t^{1+\sigma}}.$$

Proof. Let $c_\alpha = \langle f,h_\alpha\rangle$. Because of the uniform convergence of the series of (5.6), (5.8) and (5.9) we get

$$\begin{aligned}\int_0^\infty \left(e^{-tH}f(x) - f(x)\right)\,\frac{dt}{t^{1+\sigma}} &= \int_0^\infty \left(\sum_\alpha e^{-t(2|\alpha|+n)}c_\alpha h_\alpha(x) - \sum_\alpha c_\alpha h_\alpha(x)\right)\,\frac{dt}{t^{1+\sigma}} \\ &= \sum_\alpha c_\alpha h_\alpha(x)\int_0^\infty \left[e^{-t(2|\alpha|+n)} - 1\right]\,\frac{dt}{t^{1+\sigma}} \\ &= \Gamma(-\sigma)\sum_\alpha (2|\alpha|+n)^\sigma c_\alpha h_\alpha(x) \;=\; \Gamma(-\sigma)H^\sigma f(x).\end{aligned}$$

□

We have the following important Lemma whose technical proof is given at the end of this section.

Lemma 5.6. *H^σ is a continuous operator on $\mathcal{S}$.*

Lemma 5.6 together with the symmetry of H^σ on $\mathcal{S}$ (that can be easily verified via Hermite series expansions) allow us to give a distributional definition of H^σ: for $u \in \mathcal{S}'$, define $H^\sigma u \in \mathcal{S}'$ through

$$\langle H^\sigma u, f\rangle := \langle u, H^\sigma f\rangle.$$

Therefore H^σ is well defined for all functions u that are tempered distributions. In particular, u can be taken from the space L^p_N of (4.1), $1 \le p < \infty$, $N > 0$.

Recall the expression of G_t given in (4.2) and the fact that (see [5])

$$e^{-tH}1(x) = \frac{1}{(\cosh 2t)^{n/2}}\, e^{-\frac{\tanh 2t}{2}|x|^2} \le 1. \tag{5.10}$$

Define the nonnegative functions

$$F_\sigma(x,z) := \frac{1}{-\Gamma(-\sigma)}\int_0^\infty G_t(x,z)\,\frac{dt}{t^{1+\sigma}}, \qquad B_\sigma(x) := \frac{1}{\Gamma(-\sigma)}\int_0^\infty \left(e^{-tH}1(x) - 1\right)\frac{dt}{t^{1+\sigma}}. \tag{5.11}$$

Theorem 5.7. *Let f be a function in L^p_N that is $C^2(\mathcal{O})$ for some open subset $\mathcal{O} \subset \mathbb{R}^n$. Then $H^\sigma f$ is a continuous function in $\mathcal{O}$ and*

$$H^\sigma f(x) = S_\sigma f(x) + f(x)B_\sigma(x), \qquad x \in \mathcal{O},$$

where

$$S_\sigma f(x) = \int_{\mathbb{R}^n} F_\sigma(x,z)(f(x) - f(z))\,dz. \tag{5.12}$$

In (5.12) we see that H^σ is a nonlocal operator. Before giving the proof of Theorem 5.7 we establish some easy consequences.

Theorem 5.8 (Maximum principle for H^σ)**.** *Let f be a function in L^p_N that is C^2 in an open set $\mathcal{O} \subset \mathbb{R}^n$. Assume that $f \ge 0$ and $f(x_0) = 0$ for some $x_0 \in \mathcal{O}$. Then $H^\sigma f(x_0) \le 0$. Moreover, $H^\sigma f(x_0) = 0$ only when $f \equiv 0$.*

Proof. By Theorem 5.7, since $f, F_\sigma \ge 0$,

$$H^\sigma f(x_0) = \int_{\mathbb{R}^n} (f(x_0) - f(z))F_\sigma(x_0,z)\,dz + f(x_0)B_\sigma(x_0) = -\int_{\mathbb{R}^n} f(z)F_\sigma(x_0,z)\,dz \le 0.$$

If $f(z) > 0$ in some set of positive measure, then the last inequality is strict. $\square$

Corollary 5.9 (Comparison principle for H^σ)**.** *Let $f, g \in L^p_N \cap C^2(\mathcal{O})$ be such that $f \ge g$ and $f(x_0) = g(x_0)$ at some $x_0 \in \mathcal{O}$. Then $H^\sigma f(x_0) \le H^\sigma g(x_0)$. Moreover, $H^\sigma f(x_0) = H^\sigma g(x_0)$ only when $f \equiv g$.*

We devote the rest of this paper to the proofs of Lemma 5.6, Theorem 5.7, Proposition 4.1, and to complete the missing details at the end of Section 3.

Proof of Lemma 5.6. Define the first order partial differential operators

$$A_i := \frac{\partial}{\partial x_i} + x_i, \quad A_{-i} := -\frac{\partial}{\partial x_i} + x_i, \qquad i = 1, \dots, n.$$

It is well known that

$$A_i h_\alpha(x) = (2\alpha_i)^{1/2} h_{\alpha - e_i}(x), \qquad A_{-i} h_\alpha(x) = (2\alpha_i + 2)^{1/2} h_{\alpha + e_i}(x), \tag{5.13}$$

where e_i is the ith coordinate vector in $\mathbb{N}_0^n$ (see [12]). This implies that $H^\sigma f \in C^\infty$ and for all $k \in \mathbb{N}$,

$$A_{i_1} \cdots A_{i_k} H^\sigma f(x) = \sum_\alpha (2|\alpha| + n)^\sigma \langle f, h_\alpha\rangle A_{i_1} \cdots A_{i_k} h_\alpha(x), \qquad i_l = \pm 1,\ l = 1, \dots, k, \tag{5.14}$$

the series converging uniformly on $\mathbb{R}^n$. Since

$$\frac{A_i + A_{-i}}{2} = x_i, \quad \frac{A_i - A_{-i}}{2} = \frac{\partial}{\partial x_i}, \qquad i = 1, \dots, n,$$

for each multi-index $\gamma, \beta \in \mathbb{N}_0^n$ we can write $x^\gamma D^\beta = x_1^{\gamma_1} \cdots x_n^{\gamma_n} \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}}$ as a finite linear combination of operators A_i and A_{-i}. Therefore, to check that $x^\gamma D^\beta H^\sigma f \in L^\infty$, it is enough to verify that for each $k \in \mathbb{N}$, $A_{i_1} \cdots A_{i_k} H^\sigma f \in L^\infty$ where $\{i_1, \ldots, i_k\} \subset \{-1, 1\}$. The identities in (5.13) easily imply the following commutation relations for Hermite functions and thus for $f \in S$:

$$\begin{cases} A_i H^\sigma f = (H+2)^\sigma A_i f, & 1 \le i \le n; \\ A_i H^\sigma f = (H-2)^\sigma A_i f, & -n \le i \le -1. \end{cases}$$

Here $(H \pm 2)^\sigma A_i f := \sum_\alpha (2|\alpha| + n \pm 2)^\sigma \langle A_i f, h_\alpha \rangle h_\alpha$. Hence, in (5.14),

$$A_{i_1} \cdots A_{i_k} H^\sigma f = \sum_\alpha (2|\alpha| + n + 2j)^\sigma \langle g, h_\alpha \rangle h_\alpha,$$

for some $j \in \mathbb{Z}$ and $g := A_{i_1} \cdots A_{i_k} f \in \mathcal{S}$. For $m \in \mathbb{N}$ sufficiently large, as in (5.7), we have

$$\left| \sum_\alpha (2|\alpha| + n + 2j)^\sigma \langle g, h_\alpha \rangle h_\alpha(x) \right| \le \|H^m g\|_{L^2(\mathbb{R}^n)} \sum_\alpha \frac{(2|\alpha| + n + 2j)^\sigma}{(2|\alpha| + n)^m} = C \|H^m g\|_{L^2(\mathbb{R}^n)}.$$

Therefore $x^\gamma D^\beta H^\sigma f \in L^\infty$. Moreover,

$$\begin{aligned} \left| x^\gamma D^\beta H^\sigma f(x) \right| = \left| \sum c_{i,k} A_{i_1} \cdots A_{i_k} H^\sigma f(x) \right| &\le C \sum \left| (H+2j)^\sigma A_{i_1} \cdots A_{i_k} f(x) \right| \\ &\le C \left(\text{seminorms in } \mathcal{S} \text{ of } (A_{i_1} \cdots A_{i_k} f)\right) = C \left(\text{seminorms in } \mathcal{S} \text{ of } f\right). \end{aligned}$$

□

For the proof of Theorem 5.7 we need some estimates on G_t, F_σ and B_σ. First we derive some equivalent formulas for these kernels. Consider the change of parameters due to S. Meda

$$t = t(s) = \frac{1}{2} \log \frac{1+s}{1-s}, \qquad t \in (0, \infty), \ s \in (0,1), \tag{5.15}$$

that produces

$$\frac{dt}{t^{1+\sigma}} = d\mu_\sigma(s) := \frac{ds}{(1-s^2)\left(\frac{1}{2} \log \frac{1+s}{1-s}\right)^{1+\sigma}}, \qquad t \in (0, \infty), \ s \in (0,1). \tag{5.16}$$

Then the heat kernel in (4.2) can be written as

$$G_{t(s)}(x,z) = \left(\frac{1-s^2}{4\pi s}\right)^{n/2} e^{-\frac{1}{4}\left[s|x+z|^2 + \frac{1}{s}|x-z|^2\right]}, \quad s \in (0,1),$$

and, from (5.11) and (5.16),

$$F_\sigma(x,z) = \frac{1}{-\Gamma(-\sigma)} \int_0^1 G_{t(s)}(x,z)\, d\mu_\sigma(s), \quad B_\sigma(x) = \frac{1}{\Gamma(-\sigma)} \int_0^1 \left(e^{-t(s)H} 1(x) - 1\right) d\mu_\sigma(s).$$

Lemma 5.10. *For all $s \in (0,1)$ and $x, z \in \mathbb{R}^n$,*

$$G_{t(s)}(x,z) \le C \left(\frac{1-s}{s}\right)^{n/2} e^{-\frac{|x||x-z|}{C}} e^{-\frac{|x-z|^2}{Cs}}. \tag{5.17}$$

In particular,

$$G_{t(s)}(x,z) \le \frac{C}{|x-z|^n} (1-s)^{n/2} e^{-\frac{|x||x-z|}{C}} e^{-\frac{|x-z|^2}{C}} e^{-\frac{|x-z|^2}{Cs}}. \tag{5.18}$$

Proof. The second estimate in the statement follows immediately from (5.17). Note that

$$G_{t(s)}(x,z) \le C \left(\frac{1-s}{s}\right)^{n/2} e^{-\frac{|x-z|^2}{8s}} e^{-\frac{1}{8}\left[s|x+z|^2 + \frac{1}{s}|x-z|^2\right]} \le C \left(\frac{1-s}{s}\right)^{n/2} e^{-\frac{|x-z|^2}{8s}} e^{-\frac{1}{8}|x-z||x+z|}.$$

We prove the second inequality above. Assume first that $|x-z| \le |x+z|$. Then by minimizing the function $\theta(s) := \frac{s}{8}|x+z|^2 + \frac{1}{8s}|x-z|^2$ for $s \in (0,1)$ we get $e^{-\frac{1}{8}\left[s|x+z|^2 + \frac{1}{s}|x-z|^2\right]} \le e^{-\frac{1}{8}|x-z||x+z|}$. In the case $|x+z| < |x-z|$ we have $e^{-\frac{1}{8}\left[s|x+z|^2 + \frac{1}{s}|x-z|^2\right]} \le e^{-\frac{1}{8s}|x-z|^2} = e^{-\frac{1}{8s}|x-z||x-z|} \le e^{-\frac{1}{8}|x-z||x+z|}$,

for all $s\in(0,1)$. To obtain estimate (5.17) proceed as follows: if $x\cdot z>0$ then $|x+z|\geq|x|$ which gives $e^{-\frac{1}{8}|x-z||x+z|}\leq e^{-\frac{1}{8}|x||x-z|}$; if $x\cdot z\leq 0$ then $|x-z|\geq|x|$ and in this situation

$$e^{-\frac{|x-z|^2}{8s}}e^{-\frac{1}{8}|x-z||x+z|}\leq e^{-\frac{|x-z|^2}{16s}}e^{-\frac{|x||x-z|}{16s}}\leq e^{-\frac{|x-z|^2}{16s}}e^{-\frac{|x||x-z|}{16}}.$$

□

Observe in (5.16) that

$$d\mu_\sigma(s)\sim\frac{ds}{s^{1+\sigma}},\ s\sim 0,\qquad d\mu_\sigma(s)\sim\frac{ds}{(1-s)(-\log(1-s))^{1+\sigma}},\ s\sim 1. \tag{5.19}$$

Lemma 5.11. *For all $x,z\in\mathbb{R}^n$,*

$$F_\sigma(x,z)\leq\frac{C}{|x-z|^{n+2\sigma}}\,e^{-\frac{|x||x-z|}{C}}e^{-\frac{|x-z|^2}{C}}\qquad\text{and}\qquad B_\sigma(x)\leq C\left(1+|x|^{2\sigma}\right). \tag{5.20}$$

Moreover, $B_\sigma\in C^\infty(\mathbb{R}^n)$.

Proof. Estimate (5.18) gives

$$F_\sigma(x,z)\leq C\frac{e^{-\frac{|x||x-z|}{C}}}{|x-z|^n}\int_0^1(1-s)^{n/2}e^{-\frac{|x-z|^2}{Cs}}\,d\mu_\sigma(s).$$

Then (5.19) implies

$$\int_0^{1/2}(1-s)^{n/2}e^{-\frac{|x-z|^2}{Cs}}\,d\mu_\sigma(s)\leq C\int_0^{1/2}e^{-\frac{|x-z|^2}{Cs}}\frac{ds}{s^{1+\sigma}}\leq C\begin{cases}\frac{1}{|x-z|^{2\sigma}}, & \text{if } |x-z|<1;\\ e^{-\frac{|x-z|^2}{C}}, & \text{if } |x-z|\geq 1;\end{cases}$$

and

$$\int_{1/2}^1(1-s)^{n/2}e^{-\frac{|x-z|^2}{Cs}}\,d\mu_\sigma(s)\leq e^{-\frac{|x-z|^2}{C}}\int_{1/2}^1\frac{ds}{(1-s)(-\log(1-s))^{1+\sigma}}=Ce^{-\frac{|x-z|^2}{C}},$$

thus the first inequality in (5.20) follows.

Apply (5.15) in (5.10) to obtain

$$e^{-t(s)H}1(x)=\left(\frac{1-s^2}{1+s^2}\right)^{n/2}e^{-\frac{s}{1+s^2}|x|^2}. \tag{5.21}$$

Then, up to the factor $\frac{1}{-\Gamma(-\sigma)}$, we can write

$$B_\sigma(x)=\int_0^1\left[\left(\frac{1-s^2}{1+s^2}\right)^{n/2}-1\right]e^{-\frac{s}{1+s^2}|x|^2}\,d\mu_\sigma(s)+\int_0^1\left(e^{-\frac{s}{1+s^2}|x|^2}-1\right)\,d\mu_\sigma(s)=I+II.$$

To estimate I and II we use (5.19) and the Mean Value Theorem. That is,

$$|I|\leq C\int_0^{1/2}\left|\left(\frac{1-s^2}{1+s^2}\right)^{n/2}-1\right|\frac{ds}{s^{\sigma+1}}+\int_{1/2}^1 d\mu_\sigma(s)\leq C\int_0^{1/2}s^2\,\frac{ds}{s^{1+\sigma}}+C\ =\ C.$$

For II we consider two cases. Assume first that $|x|^2\leq 2$. Then

$$|II|\leq C\int_0^{1/2}\left|e^{-\frac{s}{1+s^2}|x|^2}-1\right|\,\frac{ds}{s^{1+\sigma}}+\int_{1/2}^1 d\mu_\sigma(s)\leq C\int_0^{1/2}|x|^2\,s\,\frac{ds}{s^{1+\sigma}}+C\leq C.$$

In the case $|x|^2>2$,

$$\begin{aligned}|II|&\leq|x|^2\int_0^{\frac{1}{|x|^2}}s\,\frac{ds}{s^{1+\sigma}}+\int_{\frac{1}{|x|^2}}^1 d\mu_\sigma(s)\leq|x|^2\int_0^{\frac{1}{|x|^2}}s^{-\sigma}\,ds+\int_{\frac{1}{|x|^2}}^1\frac{ds}{(1-s)\left(-\log(1-s)\right)^{1+\sigma}}\\&=C\,|x|^{2\sigma}+C\left[-\log\left(1-\frac{1}{|x|^2}\right)\right]^{-\sigma}\leq C\,|x|^{2\sigma},\end{aligned}$$

since $-\log(1-s) \sim s$ as $s \to 0$. Therefore the second fact of (5.20) follows. B_σ is differentiable since the gradient of the integrand in its definition is bounded by

$$2\,|x|\,\frac{s}{1+s^2}\left(\frac{1-s^2}{1+s^2}\right)^{n/2} e^{-\frac{s}{1+s^2}|x|^2} \leq C\,|x|\,s \in L^1((0,1);d\mu_\sigma(s)),$$

thus we can differentiate inside the integral:

$$\nabla B_\sigma(x) = 2x\int_0^1 \frac{s}{1+s^2}\left(\frac{1-s^2}{1+s^2}\right)^{n/2} e^{-\frac{s}{1+s^2}|x|^2}\;d\mu_\sigma(s).$$

For higher order derivatives we can proceed similarly. □

Proof of Theorem 5.7. Take first $f \in \mathcal{S}$. Since $e^{-tH}1(x)$ is not a constant function we write

$$\begin{aligned}
\int_0^\infty \left(e^{-tH}f(x) - f(x)\right)\,\frac{dt}{t^{1+\sigma}} &= \int_0^\infty \left(\int_{\mathbb{R}^n} G_t(x,z)f(z)\;dz - f(x)\right)\,\frac{dt}{t^{1+\sigma}} \\
&= \int_0^\infty \left[\int_{\mathbb{R}^n} G_t(x,z)(f(z)-f(x))\;dz + f(x)\left(\int_{\mathbb{R}^n} G_t(x,z)\;dz - 1\right)\right]\,\frac{dt}{t^{1+\sigma}} \\
&= \int_0^\infty \int_{\mathbb{R}^n} G_t(x,z)(f(z)-f(x))\;dz\;\frac{dt}{t^{1+\sigma}} + f(x)\int_0^\infty \left(e^{-tH}1(x) - 1\right)\frac{dt}{t^{1+\sigma}} \\
&= \int_0^1 \int_{\mathbb{R}^n} G_{t(s)}(x,z)(f(z)-f(x))\;dz\;d\mu_\sigma(s) + f(x)B_\sigma(x).
\end{aligned}$$

Due to Lemma 5.5, the first integral above is well defined and converges absolutely. Write the integral in the last line as $I_\delta + I_{\delta^c}$ with $I_{\delta^c} = \int_0^1 \int_{|x-z|>\delta} G_{t(s)}(x,z)(f(z)-f(x))\;dz\;d\mu_\sigma(s)$, for some $\delta > 0$ (in this step δ is arbitrary, but we will fix it later). Estimate (5.20) implies that I_{δ^c} is absolutely convergent and $|I_{\delta^c}| \leq C\,\|f\|_{L^\infty(\mathbb{R}^n)}$. Pass to polar coordinates in I_δ:

$$\begin{aligned}
I_\delta &= \int_0^1 \left(\frac{1-s^2}{4\pi s}\right)^{n/2} \int_{|x-z|<\delta} e^{-\frac14\left[s|x+z|^2 + \frac1s|x-z|^2\right]}(f(z)-f(x))\;dz\;d\mu_\sigma(s) \\
&= \int_0^1 \left(\frac{1-s^2}{4\pi s}\right)^{n/2} \int_0^\delta r^{n-1}e^{-\frac{r^2}{4s}} \int_{|z'|=1} e^{-\frac{s}{4}\left|2x+rz'\right|^2}(f(x+rz')-f(x))\;dS(z')drd\mu_\sigma(s).
\end{aligned}$$

To estimate $I_{S^{n-1}} := \int_{|z'|=1} e^{-\frac{s}{4}\left|2x+rz'\right|^2}(f(x+rz')-f(x))\;dS(z')$ use the Taylor expansions of f and $\psi_s(w) := e^{-\frac{s}{4}|w|^2}$ and cancel out terms:

$$\begin{aligned}
I_{S^{n-1}} &= \int_{|z'|=1} \left(e^{-\frac{s}{4}|2x|^2} + R_0\psi_s(x,rz')\right)\left(\nabla f(x)(rz') + R_1 f(x,rz')\right)\;dS(z') \\
&= \int_{|z'|=1} \left[e^{-\frac{s}{4}|2x|^2}R_1 f(x,rz') + R_0\psi_s(x,rz')\nabla f(x)(rz') + R_0\psi_s(x,rz')R_1 f(x,rz')\right]dS(z').
\end{aligned}$$

Since $|R_0\psi_s(x,rz')| \leq s^{1/2}r$ and $|R_1 f(x,rz')| \leq \left\|D^2 f\right\|_{L^\infty(B_\delta(x))} r^2$, we have $|I_{S^{n-1}}| \leq Cr^2$. Thus

$$\begin{aligned}
|I_\delta| &\leq \int_0^1 \left(\frac{1-s^2}{4\pi s}\right)^{n/2} \int_0^\delta r^{n-1}e^{-\frac{r^2}{4s}}\,|I_{S^{n-1}}|\;dr\;d\mu_\sigma(s) \\
&\leq C\int_0^\delta r^{n+1}\int_0^1 \frac{1}{s^{n/2}}\;e^{-\frac{r^2}{4s}}\;d\mu_\sigma(s)\;dr \leq C\int_0^\delta r^{n+1}\frac{1}{r^{n+2\sigma}}\;dr = C\delta^{2-2\sigma}.
\end{aligned}$$

Hence I_δ converges. The conclusion follows, for $f \in \mathcal{S}$, by Fubini's Theorem.

Now assume that $f \in L^p_N$, $1 \leq p < \infty$, $N > 0$, is a C^2 function in $\mathcal{O}$. Then $H^\sigma f$ is well defined as a tempered distribution. Fix an arbitrary $x \in \mathcal{O}$ and take $\delta > 0$ so that $B_\delta(x) \subset \mathcal{O}$. Observe that the integral in (5.12) is well defined: just apply Taylor's Theorem (as above) in I_δ and the L^p_N condition together with (5.20) in I_{δ^c}. Let $\varepsilon > 0$. There exists a sequence $f_k \in C_c^\infty(\mathbb{R}^n)$ such that $\left\|D^2 f_k\right\|_{L^\infty(B_\delta(x))} \leq \left\|D^2 f\right\|_{L^\infty(B_\delta(x))}$ for all k, f_k converges uniformly to f in $B_\delta(x)$ and $f_k \to f$ in

the norm of L^p_N, as $k\to\infty$ (use mollifiers and multiplication by a smooth cutoff function). Since B_σ is a continuous function, f_kB_σ converges uniformly to fB_σ on $B_\delta(x)$. Let $0<\rho<\delta/2$ be such that for all k

$$\left|\int_{B_\rho(x)}F_\sigma(x,z)(f_k(x)-f_k(z))\,dz\right|<\frac{\varepsilon}{3},\quad\text{and}\quad\left|\int_{B_\rho(x)}F_\sigma(x,z)(f(x)-f(z))\,dz\right|<\frac{\varepsilon}{3}.$$

For k sufficiently large, by Hölder's inequality,

$$\begin{aligned}&\left|\int_{B^c_\rho(x)}F_\sigma(x,z)(f_k(x)-f_k(z))\,dz-\int_{B^c_\rho(x)}F_\sigma(x,z)(f(x)-f(z))\,dy\right|\le\\&\le|f_k(x)-f(x)|\int_{B^c_\rho(x)}F_\sigma(x,z)\,dz+\int_{B^c_\rho(x)}F_\sigma(x,z)\,|f_k(z)-f(z)|\,dz\\&\le C\left(|f_k(x)-f(x)|+\|f_k-f\|_{L^p_N}\right)<\frac{\varepsilon}{3}.\end{aligned}$$

Thus

$$S_\sigma f_k(x)\rightrightarrows\int_{\mathbb{R}^n}F_\sigma(x,z)(f(x)-f(z))\,dz$$

in $B_\delta(x)$. But $H^\sigma f_k\to H^\sigma f$ in $\mathcal{S}'$. By uniqueness of the limits, $S_\sigma f(x)$ coincides with the integral in (5.12). Moreover, $H^\sigma f$ is continuous in $B_\delta(x)$ because it is the uniform limit of continuous functions. □

Proof of Proposition 4.1. By (5.15), (5.17) and Hölder's inequality,

$$\left|e^{-t(s)H}f(x)\right|\le\frac{C\,\|f\|_{L^p_N}}{s^{n/2}}\left(\int_{\mathbb{R}^n}e^{-\frac{p'|x-z|^2}{C}}(1+|z|^2)^{Np'}\,dz\right)^{1/p'}\le C\frac{(1+|x|^\rho)\,\|f\|_{L^p_N}}{s^{n/2}}.$$

For (4.3) note that if $0<s<\frac12$, then $s<t(s)<\frac43 s$. The equality $\partial_te^{-tH}f(x)=\int_{\mathbb{R}^n}\partial_tG_t(x,z)f(z)\,dz$ is valid if the last integral is absolutely convergent for all t in some interval. But $\partial_tG_t(x,z)f(z)=-H_xG_t(x,z)f(z)$, therefore we have to verify that $\int_{\mathbb{R}^n}H_xG_{t(s)}(x,z)f(z)\,dz$ converges absolutely for all s in some interval. This last statement is true since

$$\left|\nabla_xG_{t(s)}(x,z)\right|\le\left(\frac{1-s^2}{s}\right)^{n/2}\frac{1}{s^{1/2}}\,e^{-c\left[s|x+z|^2+\frac1s|x-z|^2\right]}$$

and

$$\left|D^2_xG_{t(s)}(x,z)\right|\le\left(\frac{1-s^2}{s}\right)^{n/2}\frac{1}{s}\,e^{-c\left[s|x+z|^2+\frac1s|x-z|^2\right]},$$

which give estimates similar to (5.17) for $\nabla G_{t(s)}$ and $D^2G_{t(s)}$. Hence $\partial_te^{-tH}f(x)=-H_xe^{-tH}f(x)$ and (4.4) follows. Observe that $t(s)\to0$ if and only if $s\to0$. For $x\in\mathcal{O}$ we have

$$\left|e^{-t(s)H}f(x)-f(x)\right|\le\left|\int_{\mathbb{R}^n}G_{t(s)}(x,z)(f(z)-f(x))\,dz\right|+|f(x)|\left|e^{-t(s)H}1(x)-1\right|.$$

The last term above tends to 0 as $t(s)\to0$ because of (5.21). Let $\delta>0$ be such that $B_\delta(x)\subset\mathcal{O}$. Then, as $f\in C^1(\overline{B_\delta(x)})$,

$$\left|\int_{B_\delta(x)}G_{t(s)}(x,z)(f(z)-f(x))\,dz\right|\le C\int_{B_\delta(x)}\frac{e^{-\frac{|x-y|^2}{Cs}}}{s^{n/2}}\,|z-x|\,dz\le C\int_{B_\delta(x)}\frac{e^{-\frac{|x-y|^2}{Cs}}}{|z-x|^{n-1}}\,dz\to0,$$

when $s \to 0$, by the Dominated Convergence Theorem. On the other hand,

$$\begin{aligned}\left|\int_{B^c_\delta(x)} G_{t(s)}(x,z)(f(z)-f(x))\,dz\right| &\le \left(\int_{B^c_\delta(x)} e^{-\frac{p|x-z|^2}{2C}}\left(\frac{|f(z)|^p}{(1+|z|^2)^{Np}}+\frac{|f(x)|^p}{(1+|z|^2)^{Np}}\right)dz\right)^{1/p}\\ &\quad\times\frac{C}{s^{n/2}}\left(\int_{B^c_\delta(x)} e^{-\frac{p'|x-z|^2}{Cs}}e^{-\frac{p'|x-z|^2}{2C}}(1+|z|^2)^{Np'}\,dz\right)^{1/p'}\\ &=: I\times II.\end{aligned}$$

Clearly $I<\infty$ and, by dominated convergence,

$$II \le C\left(\int_{B^c_\delta(x)} \frac{e^{-\frac{p'|x-z|^2}{Cs}}}{|x-z|^{np'}}\, e^{-\frac{p'|x-z|^2}{C}}(1+|z|^2)^{Np'}\,dz\right)^{1/p'} \to 0, \qquad \text{as } s\to 0.$$

□

Remark 5.12. If $f \in L^p_N \cap C^2(\mathcal{O})$ then, for each $x \in \mathcal{O}$,

$$\int_0^\infty \left|e^{-tH}f(x)-f(x)\right|\,\frac{dt}{t^{1+\sigma}} = \int_0^1 \left|e^{-t(s)H}f(x)-f(x)\right|\,d\mu_\sigma(s) < \infty.$$

Indeed, by (4.3),

$$\int_{\frac12\log 3}^\infty \left|e^{-tH}f(x)-f(x)\right|\,\frac{dt}{t^{1+\sigma}} \le C(x)\int_{\frac12\log 3}^\infty \frac{1}{t^{n/2}}\,\frac{dt}{t^{1+\sigma}} < \infty,$$

and

$$\begin{aligned}&\int_0^{\frac12\log 3} \left|e^{-tH}f(x)-f(x)\right|\,\frac{dt}{t^{1+\sigma}} = \int_0^{1/2}\left|e^{-t(s)H}f(x)-f(x)\right|\,d\mu_\sigma(s)\\ &\le C\int_0^{1/2}\left|\int_{\mathbb{R}^n} G_{t(s)}(x,z)(f(z)-f(x))dz\right|\frac{ds}{s^{1+\sigma}} + C\,|f(x)|\int_0^{1/2}\left[1-\left(\frac{1-s^2}{1+s^2}\right)^{n/2} e^{-\frac{s}{1+s^2}|x|}\right]\frac{ds}{s^{1+\sigma}}.\end{aligned}$$

Both integrals above are finite: the first one by the arguments in the proof of Theorem 5.7 (Taylor's Theorem) and the second one because of the Mean Value Theorem.

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References

[1] Caffarelli, L., Salsa, S., Silvestre, L. (2008). Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian. *Invent. Math.* 171:425–461.

[2] Caffarelli, L., Silvestre, L. (2007). An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations* 32:1245–1260.

[3] Davies, E. B. (1989). *Heat Kernels and Spectral Theory.* Cambrigde Tracts in Mathematics, Vol. 92. Cambridge: Cambridge University Press.

[4] Gutiérrez, C. E. (1989). Harnack's inequality for degenerate Schrödinger operators. *Trans. Amer. Math. Soc.* 312: 403–419.

[5] Harboure, E., de Rosa, L., Segovia, C., Torrea, J. L. (2004). L^p–dimension free boundedness for Riesz transforms associated to Hermite functions. *Math. Ann.* 328:653–682.

[6] Lebedev, N. N. (1972). *Special functions and their applications.* New York: Dover.

[7] Muckenhoupt, B., Stein, E. M. (1965). Classical expansions and their relation to conjugate harmonic functions. *Trans. Amer. Math. Soc.* 118:17–92.

[8] Rudin, W. (1973). *Functional analysis.* McGraw-Hill Series in Higher Mathematics. New York-Düsseldorf-Johannesburg: McGraw-Hill Book Company.

[9] Silvestre, L. (2005). PhD thesis. *The University of Texas at Austin.*

[10] Silvestre, L. (2007). Regularity of the obstacle problem for a fractional power of the Laplace operator. *Comm. Pure Appl. Math.* 60:67–112.

[11] Stein, E. M. (1970). *Topics in harmonic analysis related to the Littlewood-Paley theory*. Annals of Mathematics Studies Vol. 63. Princeton, NJ: Princeton Univ. Press.

[12] Thangavelu, S. (1993). *Lectures on Hermite and Laguerre Expansions*. Mathematical Notes Vol. 42. Princeton, NJ: Princeton Univ. Press.

[13] Trudinger, N. S. (1973). Linear elliptic operators with measurable coefficients. *Ann. Scuola Norm. Sup. Pisa (3)* 27:265–308.

[14] Zaslavsky, G. M. (2002). Chaos, fractional kinetics, and anomalous transport. *Phys. Rep.* 371:461–580.

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